1. [20] The group $SU(2)$ shows up in surprising places. Consider, for example, the two-dimensional harmonic oscillators, which can be written in the form

$$H = \hbar \omega (a_1^+ a_1 + a_2^+ a_2 + 1)$$

where $a_1$ and $a_2$ are two operators satisfying

$$[a_i, a_j^+] = \delta_{ij}, \quad [a_i, a_j] = [a_i^+, a_j^+] = 0$$

By conventional means, it is not hard to show that this Hamiltonian results in degenerate eigenvalues. But why? Is there a symmetry which results in this degeneracy?

(a) [8] Define the three operators

$$T_1 = \frac{i}{2} (a_1^+ a_2 + a_2^+ a_1), \quad T_2 = \frac{i}{2} (a_1^+ a_1 - a_2^+ a_2), \quad T_3 = \frac{i}{2} (a_1^+ a_1 - a_2^+ a_2).$$

Show that these operators satisfy the $SU(2)$ commutation relations,

$$[T_a, T_b] = i \sum_c \epsilon_{abc} T_c.$$

This is three relations in all.

We simply start working them out:

$$[T_1, T_2] = \frac{i}{4} \left[ a_1^+ a_2 + a_2^+ a_1, a_2^+ a_1 - a_1^+ a_2 \right] = \frac{i}{4} \left[ a_1^+ a_2 a_2^+ a_1 - a_2^+ a_1 a_2^+ a_1 \right] = \frac{i}{4} \left[ a_1^+ a_2, a_2^+ a_1 \right]$$

$$= \frac{i}{4} \left\{ a_1^+ \left[ a_2, a_1^+ a_2 \right] a_1 + a_2^+ \left[ a_1, a_1^+ a_2 \right] a_1 \right\} = \frac{i}{4} \left\{ a_1^+ a_2 a_2^+ a_1 - a_2^+ a_1 a_2^+ a_1 \right\}$$

$$= \frac{i}{4} \left\{ a_1^+ a_2 a_2^+ a_1 - a_2^+ a_1 a_2^+ a_1 \right\} = i T_3,$$

$$[T_2, T_3] = \frac{i}{4} \left[ a_1^+ a_2 - a_2^+ a_1, a_2^+ a_1 - a_1^+ a_2 \right]$$

$$= \frac{i}{4} \left\{ a_1^+ a_2 \left[ a_2, a_1^+ a_2 \right] a_1 - a_2^+ a_1 \left[ a_1, a_1^+ a_2 \right] a_1 \right\} = \frac{i}{4} \left\{ a_1^+ a_2 a_2^+ a_1 - a_2^+ a_1 a_2^+ a_1 \right\}$$

$$= \frac{i}{4} \left\{ a_1^+ a_2 a_2^+ a_1 - a_2^+ a_1 a_2^+ a_1 \right\} = i T_1,$$

$$[T_3, T_1] = \frac{i}{4} \left[ a_1^+ a_2 - a_2^+ a_1, a_1^+ a_2 + a_2^+ a_1 \right]$$

$$= \frac{i}{4} \left\{ a_1^+ a_2 \left[ a_1, a_1^+ a_2 \right] a_1 - a_2^+ a_1 \left[ a_2, a_1^+ a_2 \right] a_1 \right\} = \frac{i}{4} \left\{ a_1^+ a_2 a_2^+ a_1 - a_2^+ a_1 a_2^+ a_1 \right\}$$

$$= \frac{i}{4} \left\{ a_1^+ a_2 a_2^+ a_1 - a_2^+ a_1 a_2^+ a_1 \right\} = i T_2,$$

It was tedious, but straightforward.
(b) [8] Show that all three of the operators commute with the Hamiltonian.

Obviously, the 1 term commutes. We therefore need only consider the other two.

\[
[T_1, H] = \frac{1}{\hbar} \hbar \omega \left[ a^*_1 a_2 + a^*_2 a_1, a^*_1 a_1 + a^*_2 a_2 \right]
= \frac{1}{\hbar} \hbar \omega \left[ a^*_1 a_2, a^*_1 a_1 \right] + \left[ a^*_1 a_1, a^*_2 a_2 \right] + \left[ a^*_2 a_2, a^*_1 a_1 \right]
= \frac{1}{\hbar} \hbar \omega \left[ a^*_1 a_1 a_2 + a^*_1 a^*_2 a_2 + a^*_2 a^*_1 a_1 + a^*_2 a_2 a_1 \right]
= \frac{1}{\hbar} \hbar \omega \left[ -a^*_1 a_2 + a^*_1 a_1 a_2 + a^*_2 a^*_1 a_1 \right] = 0,
\]

\[
[T_2, H] = \frac{1}{\hbar} \hbar \omega \left[ a^*_2 a_1 - a^*_1 a_2, a^*_1 a^*_1 a_1 + a^*_2 a^*_2 a_2 \right]
= \frac{1}{\hbar} \hbar \omega \left[ a^*_2 a_1, a^*_1 a_1 \right] + \left[ a^*_2 a_2, a^*_2 a_2 \right] - \left[ a^*_1 a_2, a^*_1 a_1 \right] - \left[ a^*_1 a_2, a^*_1 a_1 \right]
= \frac{1}{\hbar} \hbar \omega \left[ a^*_1 a^*_1 a_1 + a^*_2 a^*_2 a_2 \right] a_1 - a^*_1 \left[ a^*_1 a^*_1 a_2 - a^*_1 a^*_1 a_1 \right] a_1
= \frac{1}{\hbar} \hbar \omega \left[ a^*_1 a^*_1 a_1 + a^*_2 a^*_2 a_2 \right] a_1 - a^*_1 \left[ a^*_1 a^*_1 a_2 - a^*_1 a^*_1 a_1 \right] a_1 = 0,
\]

\[
[T_3, H] = \frac{1}{\hbar} \hbar \omega \left[ a^*_1 a_1 - a^*_1 a_2, a^*_1 a^*_1 a_1 + a^*_2 a^*_2 a_2 \right] = \frac{1}{\hbar} \hbar \omega \left[ a^*_1 a^*_1 a_1 + a^*_2 a^*_2 a_2 \right] = 0.
\]

It immediately follows that we can choose, for example, eigenstates of \( H \) that are also eigenstates of \( T_3 \) and \( T^2 \), which we would write as \( |j, m\rangle \), with \( m \) running from \(-j\) to \(+j\).

At the moment, however, we have no idea what the energy of these states will be, though we do know they will be \( 2j + 1 \) degenerate.

(c) [4] There is a simple relationship between the Hamiltonian and the generators, namely

\[
T^2 = T_1^2 + T_2^2 + T_3^2 = \frac{1}{\hbar^2} \left[ \hbar^2 / \hbar \omega^2 - 1 \right]
\]

Demonstrating this is straightforward but laborious. Using this relationship, find the possible eigenvalues of \( H \), and their degeneracy, using only your knowledge of the eigenvalues of \( T^2 \).

The eigenstates will have energies \( E \). If we put the eigenstate \( |j, m\rangle \) on the right, we know the eigenvalue of \( T^2 \) will be \( j^2 + j \). We therefore have

\[
T^2 |j, m\rangle = \frac{1}{\hbar^2} \left[ \hbar^2 / \hbar^2 \omega^2 - 1 \right] |j, m\rangle,
\]

\[
(j^2 + j) |j, m\rangle = \frac{1}{\hbar^2} \left[ E^2 / \hbar^2 \omega^2 - 1 \right] |j, m\rangle,
\]

\[
4j^2 + 4j = E^2 / \hbar^2 \omega^2 - 1,
\]

\[
E^2 / \hbar^2 \omega^2 = 4j^2 + 4j + 1 = (2j + 1)^2,
\]

\[
E = (2j + 1) \hbar \omega
\]

So the energies will be positive integers times \( \hbar \omega \). The degeneracy is \( 2j + 1 \).