Physics 745 - Group Theory

Solution Set 23

1. **[20] The group $SU(2)$ shows up in surprising places. Consider, for example, the two-dimensional harmonic oscillators, which can be written in the form**

$$H = \hbar\omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)$$

where a_1 and a_2 are two operators satisfying

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \quad \left[a_i, a_j\right] = \left[a_i^\dagger, a_j^\dagger\right] = 0$$

By conventional means, it is not hard to show that this Hamiltonian results in degenerate eigenvalues. But why? Is there a symmetry which results in this degeneracy?

(a) [8] Define the three operators

$$\mathcal{T}_1 = \tfrac{1}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \quad \mathcal{T}_2 = \tfrac{i}{2}\left(a_2^\dagger a_1 - a_1^\dagger a_2\right), \quad \mathcal{T}_3 = \tfrac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right).$$

Show that these operators satisfy the $SU(2)$ commutation relations,

$$\left[\mathcal{T}_a, \mathcal{T}_b\right] = i\sum_c \varepsilon_{abc}\mathcal{T}_c\ .$$

This is three relations in all.

We simply start working them out:

$$\begin{aligned}
\left[\mathcal{T}_1, \mathcal{T}_2\right] &= \tfrac{i}{4}\left[a_1^\dagger a_2 + a_2^\dagger a_1, a_2^\dagger a_1 - a_1^\dagger a_2\right] = \tfrac{i}{4}\left\{\left[a_1^\dagger a_2, a_2^\dagger a_1\right] - \left[a_2^\dagger a_1, a_1^\dagger a_2\right]\right\} = \tfrac{i}{2}\left[a_1^\dagger a_2, a_2^\dagger a_1\right] \\
&= \tfrac{i}{2}\left\{a_1^\dagger\left[a_2, a_2^\dagger a_1\right] + \left[a_1^\dagger, a_2^\dagger a_1\right]a_2\right\} = \tfrac{i}{4}\left\{a_1^\dagger\left[a_2, a_2^\dagger\right]a_1 + a_2^\dagger\left[a_1^\dagger, a_1\right]a_2\right\} \\
&= \tfrac{i}{2}\left\{a_1^\dagger a_1 - a_2^\dagger a_2\right\} = i\mathcal{T}_3, \\
\left[\mathcal{T}_2, \mathcal{T}_3\right] &= \tfrac{i}{4}\left[a_2^\dagger a_1 - a_1^\dagger a_2, a_1^\dagger a_1 - a_2^\dagger a_2\right] \\
&= \tfrac{i}{4}\left\{\left[a_2^\dagger a_1, a_1^\dagger a_1\right] - \left[a_2^\dagger a_1, a_2^\dagger a_2\right] - \left[a_1^\dagger a_2, a_1^\dagger a_1\right] + \left[a_1^\dagger a_2, a_2^\dagger a_2\right]\right\} \\
&= \tfrac{i}{4}\left\{a_2^\dagger\left[a_1, a_1^\dagger\right]a_1 - a_2^\dagger\left[a_2^\dagger, a_2\right]a_1 - a_1^\dagger\left[a_1^\dagger, a_1\right]a_2 + a_1^\dagger\left[a_2, a_2^\dagger\right]a_2\right\} \\
&= \tfrac{i}{4}\left\{a_2^\dagger a_1 + a_2^\dagger a_1 + a_1^\dagger a_2 + a_1^\dagger a_2\right\} = i\mathcal{T}_1, \\
\left[\mathcal{T}_3, \mathcal{T}_1\right] &= \tfrac{1}{4}\left[a_1^\dagger a_1 - a_2^\dagger a_2, a_1^\dagger a_2 + a_2^\dagger a_1\right] \\
&= \tfrac{1}{4}\left\{\left[a_1^\dagger a_1, a_1^\dagger a_2\right] + \left[a_1^\dagger a_1, a_2^\dagger a_1\right] - \left[a_2^\dagger a_2, a_1^\dagger a_2\right] - \left[a_2^\dagger a_2, a_2^\dagger a_1\right]\right\} \\
&= \tfrac{1}{4}\left\{a_1^\dagger\left[a_1, a_1^\dagger\right]a_2 + a_2^\dagger\left[a_1^\dagger, a_1\right]a_1 - a_1^\dagger\left[a_2^\dagger, a_2\right]a_2 - a_2^\dagger\left[a_2, a_2^\dagger\right]a_1\right\} \\
&= \tfrac{1}{4}\left\{a_1^\dagger a_2 - a_2^\dagger a_1 + a_1^\dagger a_2 - a_2^\dagger a_1\right\} = i\mathcal{T}_2
\end{aligned}$$

It was tedious, but straightforward.

(b) [8] Show that all three of the operators commute with the Hamiltonian.

Obviously, the 1 term commutes. We therefore need only consider the other two.

$$
\begin{aligned}
\left[\mathcal{T}_1, H\right] &= \tfrac{1}{2}\hbar\omega\left[a_1^\dagger a_2 + a_2^\dagger a_1, a_1^\dagger a_1 + a_2^\dagger a_2\right] \\
&= \tfrac{1}{2}\hbar\omega\left\{\left[a_1^\dagger a_2, a_1^\dagger a_1\right] + \left[a_1^\dagger a_2, a_2^\dagger a_2\right] + \left[a_2^\dagger a_1, a_1^\dagger a_1\right] + \left[a_2^\dagger a_1, a_2^\dagger a_2\right]\right\} \\
&= \tfrac{1}{2}\hbar\omega\left\{a_1^\dagger\left[a_1^\dagger, a_1\right]a_2 + a_1^\dagger\left[a_2, a_2^\dagger\right]a_2 + a_2^\dagger\left[a_1, a_1^\dagger\right]a_1 + a_2^\dagger\left[a_2^\dagger, a_2\right]a_1\right\} \\
&= \tfrac{1}{2}\hbar\omega\left\{-a_1^\dagger a_2 + a_1^\dagger a_2 + a_2^\dagger a_1 - a_2^\dagger a_1\right\} = 0, \\
\left[\mathcal{T}_2, H\right] &= \tfrac{i}{2}\hbar\omega\left[a_2^\dagger a_1 - a_1^\dagger a_2, a_1^\dagger a_1 + a_2^\dagger a_2\right] \\
&= \tfrac{i}{2}\hbar\omega\left\{\left[a_2^\dagger a_1, a_1^\dagger a_1\right] + \left[a_2^\dagger a_1, a_2^\dagger a_2\right] - \left[a_1^\dagger a_2, a_1^\dagger a_1\right] - \left[a_1^\dagger a_2, a_2^\dagger a_2\right]\right\} \\
&= \tfrac{i}{2}\hbar\omega\left\{a_2^\dagger\left[a_1, a_1^\dagger\right]a_1 + a_2^\dagger\left[a_2^\dagger, a_2\right]a_1 - a_1^\dagger\left[a_1^\dagger, a_1\right]a_2 - a_1^\dagger\left[a_2, a_2^\dagger\right]a_2\right\} \\
&= \tfrac{i}{2}\hbar\omega\left\{a_2^\dagger a_1 - a_2^\dagger a_1 + a_1^\dagger a_2 - a_1^\dagger a_2\right\} = 0, \\
\left[\mathcal{T}_3, H\right] &= \tfrac{1}{2}\hbar\omega\left[a_1^\dagger a_1 - a_2^\dagger a_2, a_1^\dagger a_1 + a_2^\dagger a_2\right] = \tfrac{1}{2}\hbar\omega\left\{\left[a_1^\dagger a_1, a_2^\dagger a_2\right] - \left[a_2^\dagger a_2, a_1^\dagger a_1\right]\right\} = 0.
\end{aligned}
$$

It immediately follows that we can choose, for example, eigenstates of H that are also eigenstates of $\mathcal{T}_3$ and $\mathcal{T}^2$, which we would write as $\left|j,m\right\rangle$, with m running from $-j$ to $+j$. At the moment, however, we have no idea what the energy of these states will be, though we *do* know they will be $2j + 1$ degenerate.

(c) [4] There is a simple relationship between the Hamiltonian and the generators, namely

$$\mathcal{T}^2 = \mathcal{T}_1^2 + \mathcal{T}_2^2 + \mathcal{T}_3^2 = \tfrac{1}{4}\left[H^2/\hbar^2\omega^2 - 1\right]$$

Demonstrating this is straightforward but laborious. Using this relationship, find the possible eigenvalues of H, and their degeneracy, using only your knowledge of the eigenvalues of $\mathcal{T}^2$.

The eigenstates will have energies E. If we put the eigenstate $\left|j,m\right\rangle$ on the right, we know the eigenvalue of $\mathcal{T}^2$ will be $j^2 + j$. We therefore have

$$
\begin{aligned}
\mathcal{T}^2\left|j,m\right\rangle &= \tfrac{1}{4}\left[H^2/\hbar^2\omega^2 - 1\right]\left|j,m\right\rangle, \\
\left(j^2 + j\right)\left|j,m\right\rangle &= \tfrac{1}{4}\left[E^2/\hbar^2\omega^2 - 1\right]\left|j,m\right\rangle, \\
4j^2 + 4j &= E^2/\hbar^2\omega^2 - 1, \\
E^2/\hbar^2\omega^2 &= 4j^2 + 4j + 1 = \left(2j+1\right)^2, \\
E &= \left(2j+1\right)\hbar\omega
\end{aligned}
$$

So the energies will be positive integers times $\hbar\omega$. The degeneracy is $2j + 1$.