Physics 745 - Group Theory

Solution Set 25

1. [15] In the electric dipole approximation, the rate at which an atom decaying from one state to another by the emission of a photon is given by $\Gamma(I \to F) = \frac{4}{3}\alpha\omega_{IF}^3 \left|\mathbf{r}_{FI}\right|^2 / c^2$ where

$$\mathbf{r}_{FI} = \langle F|\mathbf{r}|I\rangle$$

The absolute value symbol means that $\left|\mathbf{r}_{FI}\right|^2$ contains not only a sum of the three components of r, but also the real and imaginary parts.

(a) [4] Demonstrate first that

$$\left|\mathbf{r}_{FI}\right|^2 = \sum_{q=-1}^{1}\left|\langle F|r_q^{(1)}|I\rangle\right|^2$$

where $r_q^{(1)}$ are the three components of the spherical tensor corresponding to the vector operator r.

This is straightforward. We have

$$\begin{aligned}
\sum_{q=-1}^{1}\left|\langle F|r_q^{(1)}|I\rangle\right|^2 &= \left|\langle F|r_{-1}^{(1)}|I\rangle\right|^2 + \left|\langle F|r_0^{(1)}|I\rangle\right|^2 + \left|\langle F|r_{+1}^{(1)}|I\rangle\right|^2 \\
&= \tfrac{1}{2}\left|\langle F|r_1|I\rangle - i\langle F|r_2|I\rangle\right|^2 + \left|\langle F|r_3|I\rangle\right|^2 + \tfrac{1}{2}\left|-\langle F|r_1|I\rangle - i\langle F|r_2|I\rangle\right|^2 \\
&= \tfrac{1}{2}\left|\langle F|r_1|I\rangle\right|^2 + \tfrac{1}{2}\left|\langle F|r_2|I\rangle\right|^2 + \left|\langle F|r_3|I\rangle\right|^2 + \tfrac{1}{2}\left|\langle F|r_1|I\rangle\right|^2 + \tfrac{1}{2}\left|\langle F|r_2|I\rangle\right|^2 \\
&= \left|\langle F|\mathbf{r}|I\rangle\right|^2 = \left|\mathbf{r}_{FI}\right|^2
\end{aligned}$$

(b) [4] An atom in the state $|njm\rangle$ with j = 3/2 is about to decay via dipole radiation. What possible j' values might be allowed for the final state $|n'j'm'\rangle$?

According to the Wigner Eckart theorem, the matrix elements we need will be of the form

$$\langle n'j'm'|r_q^{(1)}|n\tfrac{3}{2}m\rangle = \frac{\langle 1\tfrac{3}{2};qm|j'm'\rangle}{\sqrt{2j'+1}}\langle n'j'\|r\|nj\rangle$$

For this rate to be non-vanishing, j' must be in the range from $\frac{3}{2}+1,\ldots,\left|\frac{3}{2}-1\right| = \frac{5}{2},\frac{3}{2},\frac{1}{2}$, so these are the three possibilities.

(c) [7] The atom is actually going to decay to a state with *j'* = ½. Using the Wigner Eckart Theorem, find the *relative* rate of decay

$$\Gamma\left(njm \to n'j'm'\right)$$

for all non-vanishing possible values of *m* and *m'*.

We need the Clebsch-Gordan coefficients $\left\langle 1\tfrac{3}{2};qm\middle|\tfrac{1}{2}m'\right\rangle$. The Clebsch routine from the web is happy to help us out. The only non-zero ones will be when $m' = q + m$, so these work out to six possibilities total. We calculate them using commands like this:

```
> for q from -1 to 1 do clebsch(1,3/2,q,1/2-q,1/2,1/2);
  clebsch(1,3/2,q,-1/2-q,1/2,-1/2); end do;
```

$$\left\langle 1\tfrac{3}{2};-1,\tfrac{3}{2}\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \left\langle 1\tfrac{3}{2};1,-\tfrac{3}{2}\middle|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{2}},$$
$$\left\langle 1\tfrac{3}{2};1,-\tfrac{1}{2}\middle|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \left\langle 1\tfrac{3}{2};-1,\tfrac{1}{2}\middle|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{6}}$$
$$\left\langle 1\tfrac{3}{2};0,\tfrac{1}{2}\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \left\langle 1\tfrac{3}{2};0,-\tfrac{1}{2}\middle|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}$$

The probabilities for these various rates is therefore proportional to

$$\Gamma\left(\tfrac{3}{2}\tfrac{3}{2} \to \tfrac{1}{2}\tfrac{1}{2}\right), \Gamma\left(\tfrac{3}{2},-\tfrac{3}{2} \to \tfrac{1}{2},-\tfrac{1}{2}\right) \propto \tfrac{1}{2},$$
$$\Gamma\left(\tfrac{3}{2}\tfrac{1}{2} \to \tfrac{1}{2}\tfrac{1}{2}\right), \Gamma\left(\tfrac{3}{2},-\tfrac{1}{2} \to \tfrac{1}{2},-\tfrac{1}{2}\right) \propto \tfrac{1}{3},$$
$$\Gamma\left(\tfrac{3}{2},-\tfrac{1}{2} \to \tfrac{1}{2},\tfrac{1}{2}\right), \Gamma\left(\tfrac{3}{2}\tfrac{1}{2} \to \tfrac{1}{2},-\tfrac{1}{2}\right) \propto \tfrac{1}{6}.$$

Or, doubling everything, another way to put this would be in a little probability table like the one at right:

$\lvert\psi_I\rangle =$	$\lvert\tfrac{3}{2},\tfrac{3}{2}\rangle$	$\lvert\tfrac{3}{2},\tfrac{1}{2}\rangle$	$\lvert\tfrac{3}{2},-\tfrac{1}{2}\rangle$	$\lvert\tfrac{3}{2},-\tfrac{3}{2}\rangle$
$\lvert\psi_F\rangle = \lvert\tfrac{1}{2}\tfrac{1}{2}\rangle$	1	$\tfrac{2}{3}$	$\tfrac{1}{3}$	0
$\lvert\psi_F\rangle = \lvert\tfrac{1}{2},-\tfrac{1}{2}\rangle$	0	$\tfrac{1}{3}$	$\tfrac{2}{3}$	1