Physics 745 - Group Theory

Solution Set 27

1. **[5] This problem has to do with demonstrating that according the isospin symmetry, the three $|\Sigma\rangle$'s all have the same mass.**

   (a) **[2] Work out the effects of the isospin operators $\mathcal{I}_\pm$ on all three of the $|\Sigma\rangle$ states.**

   These are in an isospin 1 state, so we have

$$\mathcal{I}_+|\Sigma^-\rangle = \sqrt{2}|\Sigma^0\rangle, \quad \mathcal{I}_+|\Sigma^0\rangle = \sqrt{2}|\Sigma^+\rangle, \quad \mathcal{I}_+|\Sigma^+\rangle = 0,$$
$$\mathcal{I}_-|\Sigma^+\rangle = \sqrt{2}|\Sigma^0\rangle, \quad \mathcal{I}_-|\Sigma^0\rangle = \sqrt{2}|\Sigma^-\rangle, \quad \mathcal{I}_-|\Sigma^-\rangle = 0.$$

   (b) **[3] Assuming that isospin commutes with the mass portion of the Hamiltonian, show that all three of the $|\Sigma\rangle$'s have the same mass.**

$$\langle\Sigma^+|H|\Sigma^+\rangle = \tfrac{1}{\sqrt{2}}\langle\Sigma^+|H\mathcal{I}_+|\Sigma^0\rangle = \tfrac{1}{\sqrt{2}}\langle\Sigma^+|\mathcal{I}_+H|\Sigma^0\rangle = \langle\Sigma^0|H|\Sigma^0\rangle$$
$$= \tfrac{1}{\sqrt{2}}\langle\Sigma^0|H\mathcal{I}_+|\Sigma^-\rangle = \tfrac{1}{\sqrt{2}}\langle\Sigma^0|\mathcal{I}_+H|\Sigma^-\rangle = \langle\Sigma^-|H|\Sigma^-\rangle,$$
$$m_{\Sigma^+} = m_{\Sigma^0} = m_{\Sigma^-}.$$

2. **[10] There is an excited set of two baryons $|N^+\rangle$ and $|N^0\rangle$ that have the same isospin properties of the neutron and proton. They can decay, in principle, into a $|\Delta\pi\rangle$ combination of two particles.**

   (a) **[4] Suppose that the Hamiltonian that performs this transition takes the form**

$$H|N^+\rangle = a|\Delta^{++};\pi^-\rangle + b|\Delta^+;\pi^0\rangle + c|\Delta^0;\pi^+\rangle$$

   Find the relative sizes of the factors *a, b,* and *c*. I recommend doing this by letting $\mathcal{I}_+$ act on both sides.

   Following the suggestion, we have

$$\mathcal{I}_+H|N^+\rangle = a\mathcal{I}_+|\Delta^{++};\pi^-\rangle + b\mathcal{I}_+|\Delta^+;\pi^0\rangle + c\mathcal{I}_+|\Delta^0;\pi^+\rangle,$$
$$0 = \sqrt{2}a|\Delta^{++};\pi^0\rangle + \sqrt{3}b|\Delta^{++};\pi^0\rangle + \sqrt{2}b|\Delta^+;\pi^+\rangle + 2c|\Delta^+;\pi^+\rangle,$$
$$0 = \left(\sqrt{2}a + \sqrt{3}b\right)|\Delta^{++};\pi^0\rangle + \left(\sqrt{2}b + 2c\right)|\Delta^+;\pi^+\rangle$$

The coefficients of each of these states must vanish, so we find

$$a = -\sqrt{\tfrac{3}{2}}b \quad \text{and} \quad c = -\sqrt{\tfrac{1}{2}}b$$

so that we have

$$H\left|N^+\right\rangle = -\sqrt{\tfrac{3}{2}}b\left|\Delta^{++};\pi^-\right\rangle + b\left|\Delta^+;\pi^0\right\rangle - b\sqrt{\tfrac{1}{2}}\left|\Delta^0;\pi^+\right\rangle$$

(b) [2] Calculate the relative rate for the decay rates $\Gamma\left(N^+ \to \Delta^{++}\pi^-\right)$, $\Gamma\left(N^+ \to \Delta^+\pi^0\right)$, and $\Gamma\left(N^+ \to \Delta^0\pi^+\right)$.

As usual, we simply have

$$\Gamma\left(N^+ \to \Delta^{++}\pi^-\right) \propto \tfrac{3}{2}b^2, \quad \Gamma\left(N^+ \to \Delta^+\pi^0\right) \propto b^2, \quad \Gamma\left(N^+ \to \Delta^0\pi^+\right) \propto \tfrac{1}{2}b^2$$

So they are in the ratio 3:2:1.

(c) [4] Now I want you to figure out how the N^0 decays. Using isospin symmetry, determine that the same interaction discussed in part (a) also leads to three decay processes for the N^0. Find the relative probability amplitudes for these processes, and predict the corresponding decay rates, and show how they relate to those found in part (b).

We start by letting $\mathcal{I}_-$ act on both sides, to yield

$$\begin{aligned}
\mathcal{I}_-H\left|N^+\right\rangle &= -\sqrt{\tfrac{3}{2}}b\mathcal{I}_-\left|\Delta^{++};\pi^-\right\rangle + b\mathcal{I}_-\left|\Delta^+;\pi^0\right\rangle - b\sqrt{\tfrac{1}{2}}\mathcal{I}_-\left|\Delta^0;\pi^+\right\rangle, \\
H\left|N^0\right\rangle &= -\sqrt{\tfrac{3}{2}}\sqrt{3}b\left|\Delta^+;\pi^-\right\rangle + \sqrt{2}b\left|\Delta^+;\pi^-\right\rangle + 2b\left|\Delta^0;\pi^0\right\rangle - b\sqrt{\tfrac{1}{2}}\sqrt{2}\left|\Delta^0;\pi^0\right\rangle \\
&\quad -b\sqrt{\tfrac{1}{2}}\sqrt{3}\left|\Delta^-;\pi^+\right\rangle \\
&= -\sqrt{\tfrac{1}{2}}b\left|\Delta^+;\pi^-\right\rangle + b\left|\Delta^0;\pi^0\right\rangle - \sqrt{\tfrac{3}{2}}b\left|\Delta^-;\pi^+\right\rangle
\end{aligned}$$

and the rates come out to

$$\Gamma\left(N^0 \to \Delta^+\pi^-\right) \propto \tfrac{1}{2}b^2, \quad \Gamma\left(N^0 \to \Delta^0\pi^0\right) \propto b^2, \quad \Gamma\left(N^0 \to \Delta^-\pi^+\right) \propto \tfrac{3}{2}b^2$$

The ratios are 1:2:3 this time, or putting it all together, we have

$$\begin{aligned}
&\frac{\Gamma\left(N^+ \to \Delta^{++}\pi^-\right)}{3} = \frac{\Gamma\left(N^+ \to \Delta^+\pi^0\right)}{2} = \Gamma\left(N^+ \to \Delta^0\pi^+\right) \\
&= \Gamma\left(N^0 \to \Delta^+\pi^-\right) = \frac{\Gamma\left(N^0 \to \Delta^0\pi^0\right)}{2} = \frac{\Gamma\left(N^0 \to \Delta^-\pi^+\right)}{3}
\end{aligned}$$