1. [15] The group SU(3) contains the group SU(2) as a subgroup, and in more than one way
   (a) [7] Show that the generators $T_1$, $T_2$ and $T_3$ form an SU(2) subgroup; that is, show that $[T_i, T_j] = i T_k$, etc. To save time, only do two of the three commutators. How does the 3 representation of SU(3) break into representations under this subgroup?

   We simply work out the commutators directly:

   $[T_1, T_2] = \frac{1}{4} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} = iT_3,$

   $[T_2, T_3] = \frac{1}{4} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iT_2,$

   $[T_3, T_1] = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iT_2.$

   That was boring. Now, technically, we don’t actually have to do anything to demonstrate this, because the matrices are already block diagonal. But in general, we would find the eigenvalues of $T_3$, which can be read off directly from this diagonal matrix, and the eigenvalues are $\{ \frac{1}{2}, -\frac{1}{2}, 0 \}$. The highest weight is $\frac{1}{2}$, which tells us we have the $\left( \frac{1}{2} \right)$ representation, which accounts for the weights $\pm \frac{1}{2}$. This leaves the weight 0, which corresponds to the (0) representation, so

   $$3 \to \left( \frac{1}{2} \right) \oplus (0)$$
(b) [8] Show that the generators $2T_2, 2T_5, 2T_7$ form an SU(2) subgroup; that is, show that $[2T_2, 2T_5] = i2T_7$, etc. To save time, only do two of the three commutators. How does the 3 representation of SU(3) break into representations under this subgroup?

We start exactly the same way, doing the commutation relations.

\[
[2T_2, 2T_5] = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = i2T_7,
\]

\[
[2T_2, 2T_7] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = i2T_5,
\]

\[
[2T_5, 2T_7] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = i2T_3.
\]

It worked! This time, though, we need to work a bit harder to get the eigenvalues of “$T_3$”, which in this context is $2T_7$. For lack of imagination, we simply take the corresponding determinant, which is

\[
0 = \det(2T_7 - \lambda I) = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -i \\ 0 & i & -\lambda \end{vmatrix} = -\lambda^3 + \lambda = -\lambda(\lambda^2 - 1)
\]

The eigenvalues are therefore \(\{1, -1, 0\}\) which is exactly the weights of the (1) representation.

\[3 \rightarrow (1)\]
2. [10] Of the eight generators, two of them can be diagonalized simultaneously (normally chosen as $T_3$ and $T_8$). In this problem, you will organize the others into pairs, comparable to the “raising” and “lowering” operators for SU(2).

(a) [5] Combine the remaining six generators, such that the commutation relations of the resulting combinations with $T_3$ and $T_8$ always come out proportional to the resulting generators. Here is one of them done for you:

$$T_A = T_1 + iT_2,$$

then

$$[T_3, T_A] = +1T_A \quad \text{and} \quad [T_8, T_A] = 0T_A$$

We can pretty much guess how we want to define all of these things:

$$T_B = T_1 - iT_2, \quad T_C = T_4 + iT_5, \quad T_D = T_4 - iT_5, \quad T_E = T_6 + iT_7, \quad T_F = T_6 - iT_7$$

It is then straightforward to work out the ten commutators:

$$[T_3, T_B] = -T_B, \quad [T_3, T_C] = \frac{1}{2}T_C, \quad [T_3, T_D] = -\frac{1}{2}T_D, \quad [T_3, T_E] = -\frac{1}{2}T_E, \quad [T_3, T_F] = \frac{1}{2}T_F,$$

$$[T_8, T_B] = 0T_B, \quad [T_8, T_C] = \frac{\sqrt{6}}{2}T_C, \quad [T_8, T_D] = -\frac{\sqrt{6}}{2}T_D, \quad [T_8, T_E] = \frac{\sqrt{6}}{2}T_E, \quad [T_8, T_F] = -\frac{\sqrt{6}}{2}T_F.$$

(b) [5] For each of the six generators you just worked out, plot on a 2D graph the resulting coefficients when you commute with $T_3$ and $T_8$. The first one is done for you. This diagram is called a root diagram.

(comment: technically, a root diagram would also include two zero roots, corresponding to the two generators $T_3$ and $T_8$ themselves, which commute with each other)

The sketch appears at right. The six points form a perfect regular hexagon. If the double root at zero is added, it would be the same as the weights of the 8 representation, which is not a coincidence.