Physics 745 - Group Theory
Solution Set 33

1. The group SO(5) has the Dynkin diagram sketched at right. 
   The shorter root can be chosen to be \( s = (0,1) \).

   (a) What is the length of the longer root \( r \)? Give the coordinates of \( r \).

   The longer root will be \( \sqrt{2} \) longer than \( s \), and it must be at \( 135^\circ \) angle compared to it. Therefore, the longer root will be of length \( \sqrt{2} \) and have coordinates
   \[
   r = (1, -1).
   \]

   (b) Use the rules described in class to determine for what positive integers \( n \) the quantities \( r + ns \) and \( s + nr \) are roots. Write them all out in coordinates.

   We can only add simple roots, we can’t subtract them. Since \( 2r \cdot s/r^2 = -1 \), we can only add \( r \) to \( s \) once. Since \( 2r \cdot s/s^2 = -2 \), we can add \( s \) to \( r \) twice. This yields two new roots, namely,
   \[
   r + s = s + r = (1, 0) \quad \text{and} \quad r + 2s = (1, 1).
   \]

   (c) Prove or disprove: More roots can be found by adding \( r \) or \( s \) to the positive roots we have already found.

   We know we can add neither \( r \) nor \( s \) to \( r + s \). We know we can’t add \( s \) to \( r + 2s \).

   Can we add \( r \) to it?

   \[
   2(r + 2s) \cdot r/r^2 = 2(1,1) \cdot (1,-1)/2 = 0.
   \]

   No, we can’t add any more.

   (d) You have found all the positive roots. Find all the negative roots. Find all the zero roots. Make a root diagram. It should be a nice, symmetric pattern.

   The negative roots are the negatives of the positive roots, or
   \[
   \{(0,-1),(-1,1),(-1,0),(-1,-1)\}
   \]

   There are also two zero roots. These are all plotted in the root diagram above.