

Solution Set 5 – Tinkham 3-1

a) There are five character classes, and hence five inequivalent irreducible representations. The sums of the squares of the dimensions must equal eight, which can only be achieved if four of them are 1-dimensional, and the other one is 2-dimensional, which lets us fill in the first column of our table

b) The trivial irrep goes in the first row. For each of the three four-element normal subgroups, we get a two-element quotient group. If you think about it, you can map the “identity” subgroup to the **1** matrix, and the “other” element to the **-1** matrix. This gives us four of our representations immediately.

D_4	E	G	AB	CD	FH
Γ^1	1	1	1	1	1
Γ^2	1	1	1	-1	-1
Γ^3	1	1	-1	1	-1
Γ^4	1	1	-1	-1	1
Γ^5	2	-2	0	0	0

The remaining one can be found by thinking of the matrices as rotations in 2D space, in which case you can show that G is the -1 matrix with trace -2, and all the others are traceless. You can check your results with various orthogonality conditions; for example, every column is orthogonal to every other column.

To check it, it is recommended that we take a row and multiply two columns together, and see if the result is equal to the correct resulting products. Hence, for example, if you take the column for AB and multiply it by the column for CD, and then times 4 (= 2 times 2, the number of elements in each column) you should get 4 times the FH column, times the dimensionality, where the four represents two (because there are two copies of FH) times two (because there are two elements in FH). For any class times the identity, this becomes the statement $\chi(E)n_i\chi(C_i) = l n_i \chi(C_i)$, and since $\chi(E) = l$, this is trivially satisfied. If you simplify the resulting expressions, you find you have to check the following for each row:

$$\begin{array}{ll}
 \chi(G)\chi(G) = l\chi(E) & \chi(AB)\chi(FH) = l\chi(CD) \\
 \chi(G)\chi(AB) = l\chi(AB) & \chi(CD)\chi(FH) = l\chi(AB) \\
 \chi(G)\chi(CD) = l\chi(CD) & 2\chi(AB)\chi(AB) = l\chi(E) + l\chi(G) \\
 \chi(G)\chi(FH) = l\chi(FH) & 2\chi(CD)\chi(CD) = l\chi(E) + l\chi(G) \\
 \chi(AB)\chi(CD) = l\chi(FH) & 2\chi(FH)\chi(FH) = l\chi(E) + l\chi(G)
 \end{array}$$

These aren't too hard to check by inspection.