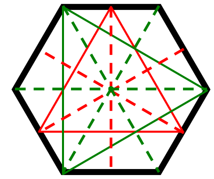
Solution Set 9

First we are supposed to demonstrate that $\mathcal{D}_3$ is a subgroup of $\mathcal{D}_{6,}$ where the former is the symmetry group of proper rotations of an equilateral triangle, and the latter is the group for a regular hexagon. Consider, for example, the hexagon sketched at right, and the red triangle inscribed inside it. Any way you rotate the triangle so it lands on itself will also cause the hexagon to land on itself. Similarly, any way you rotate the green triangle to land on itself will cause the hexagon to land on itself. In conclusion, we now have *two* proofs that $\mathcal{D}_3$ is a subgroup of $\mathcal{D}_6$.



One thing that is interesting about this is that the two different ways of choosing subgroups are inequivalent. For example, the red dashed lines are C_2 axes for the red triangle, but not the green triangle, while the green dashed lines are C_2 axes for the green triangle, but not the red. Hence we've actually proven that $\mathcal{D}_6$ has *two distinct* $\mathcal{D}_3$ subgroups. This will cause us not very significant problems later.

Now, we need to figure out what the elements of $\mathcal{D}_3$ correspond to in $\mathcal{D}_6$. Obviously, E corresponds to E, and since any C_3 rotation is still a C_3 rotation, that's pretty clear as well. But what about C_2? By the conventions of Tinkham, without a prime, C_2 always means a rotation around the principal axis (*z*-axis), whereas C_2' or C_2" means a rotation around a perpendicular axis. It makes sense, for example, to assume that any of the red axes in the figure above correspond to C_2', while any of the green axes is C_2". So what does C_2' in $\mathcal{D}_3$ correspond to in the larger group? Well, if the subgroup is the symmetry of the red triangle, then it is the red dashed lines, which are the C_2' axes, while if it is the green triangle, then it is the green dashed lines, which are the C_2" axes. Either one is fine, and both are possible. We can pick whichever one we prefer, so to make it simple, I'll choose that the C_2' axes of $\mathcal{D}_3$ correspond to the C_2' axes of $\mathcal{D}_6$.

Now we make the character table for $\mathcal{D}_3$, and add below it character tables for all the irreducible representations (irreps) of $\mathcal{D}_6$. The table appears at right. For the rows of the representations of $\mathcal{D}_6$, we have to find combinations of the first three rows whose totals are the rows below. The results are pretty easy to do by inspection, since every one of the new rows is identical with one of the rows above it, so we see that the "breakdown" of the irreps of the big group under the smaller group is

$A_1, B_1 \rightarrow A_1 \quad A_2, B_2 \rightarrow A_2 \quad E_1, E_2 \rightarrow E$

where in every case the irreps on the left are of $\mathcal{D}_6$, and on the right of $\mathcal{D}_3$. There is no "splitting" caused by the reduced symmetry.

	E	$2C_3$	$3C_2$'
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0
A_1	1	1	1
A_2	1	1	-1
B_1	1	1	1
B_2	1	1	-1
E_1	2	-1	0
E_2	2	-1	0

It is interesting that in this case nothing interesting seems to be happening. The reason is that $\mathcal{D}_6 = \mathcal{D}_3 \times \mathcal{C}_2$, so we are just dealing with a simple product group, and since the group being multiplied ($\mathcal{C}_2$) is a very simple group with only one-dimensional irreps, the irreps of $\mathcal{D}_6$ are no bigger than those of $\mathcal{D}_3$.