Physics 745 - Group Theory

Solution to Final, Spring 2009

You may use (1) class notes, (2) the text, (3) former homeworks and solutions (available online), or (4) any math references, such as integral tables, Maple, etc, including any routines I provided for you. If you cannot solve an equation, try to go on, as if you knew the answer. Feel free to contact me with questions. Note that the final problem is more like a homework problem, you should feel free to ask me for help on it. Each question is worth 20 points out of a total of 100 points.

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1. The proper icosohedral group is a 60 element group. Its character table is given at right. The irreps have been unimaginatively labeled as *A, B, C, D,* and *E*. For each of the five tensor products with *E*, break the tensor product into appropriate irreps; *i.e.*, work out $A\otimes E$, $B\otimes E$, … , $E\otimes E$.

$\mathcal{I}$	E	$20C_3$	$12C_5$	$12C_5^2$	$15C_2$
A	1	1	1	1	1
B	3	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1
C	3	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1
D	4	1	-1	-1	0
E	5	-1	0	0	1
$A\otimes E$	5	-1	0	0	1
$B\otimes E$	15	0	0	0	-1
$C\otimes E$	15	0	0	0	-1
$D\otimes E$	20	-1	0	0	0
$E\otimes E$	25	1	0	0	1

The table has been extended to include the characters for these tensor products. The character of the tensor product is just the product of the characters for the individual irreps. Obviously, $A\otimes E = E$, but the others are not as obvious. For $B\otimes E$ or $C\otimes E$, the number of contributions like *E* is $\left[1\cdot 15\cdot 5+15\cdot(-1)\cdot 1\right]/60=1$, and the number of copies of *B* or *C* is $\left[1\cdot 15\cdot 3+15\cdot(-1)\cdot(-1)\right]/60=1$, and the number of copies of *D* is $\left[1\cdot 15\cdot 4\right]/60=1$. This accounts for all 15 dimensions. For $D\otimes E$, we have $\left[1\cdot 20\cdot 5+20\cdot(-1)\cdot(-1)\right]/60=2$ copies of *E*, $\left[1\cdot 20\cdot 4+20\cdot(-1)\cdot 1\right]/60=1$ copies of *D*, and $\left[1\cdot 20\cdot 3\right]/60=1$ copies of *B* or *C*. Finally, for $E\otimes E$, we have $\left[1\cdot 25\cdot 4+20\cdot 1\cdot 1\right]/60=2$ copies of *D*, $\left[1\cdot 25\cdot 5+20\cdot 1\cdot(-1)+15\cdot 1\cdot 1\right]/60=2$ copies of *E*, $\left[1\cdot 25\cdot 3+15\cdot 1\cdot(-1)\right]/60=1$ copy of *B* or *C*, and $\left[1\cdot 25\cdot 1+20\cdot 1\cdot 1+15\cdot 1\cdot 1\right]/60=1$ copy of *A*. In summary, we have

$$\begin{aligned}
A\otimes E &= E,\\
B\otimes E &= C\otimes E = B\oplus C\oplus D\oplus E,\\
D\otimes E &= B\oplus C\oplus D\oplus E\oplus E,\\
E\otimes E &= A\oplus B\oplus C\oplus D\oplus D\oplus E\oplus E.
\end{aligned}$$

2. The group SO(5) has ten generators, which can be labeled T_{ab}, where $a,b \in \{1,2,3,4,5\}$, and $a \neq b$ (they are defined in such a way that $T_{ab} = -T_{ba}$, which is why there are only ten of them). They satisfy the commutation relations

$$[T_{ab}, T_{cd}] = i\left(\delta_{ac}T_{bd} + \delta_{bd}T_{ac} - \delta_{ad}T_{bc} - \delta_{bc}T_{ad}\right)$$

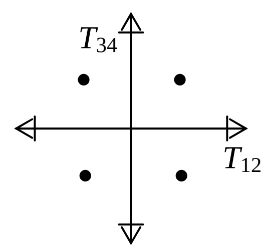
where δ_{ab} is the Kronecker delta function.

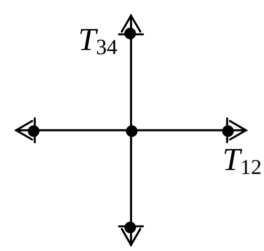
(a) Let $J_1 = T_{13}$, $J_2 = T_{23}$, $J_3 = T_{12}$. Show that J generates an SU(2) subgroup of SO(5), *i.e.*, show all three of the commutators $[J_a, J_b] = i\varepsilon_{abc}J_c$.

Let's just work them all out. We have

$$\begin{aligned}
[J_1, J_2] &= [T_{13}, T_{23}] = i\left(\delta_{12}T_{33} + \delta_{33}T_{12} - \delta_{13}T_{32} - \delta_{32}T_{13}\right) = iT_{12} = iJ_3, \\
[J_2, J_3] &= [T_{23}, T_{12}] = i\left(\delta_{21}T_{32} + \delta_{32}T_{21} - \delta_{22}T_{31} - \delta_{31}T_{22}\right) = -iT_{31} = iT_{31} = iJ_1, \\
[J_3, J_1] &= [T_{12}, T_{13}] = i\left(\delta_{11}T_{23} + \delta_{23}T_{23} - \delta_{13}T_{21} - \delta_{21}T_{13}\right) = iT_{23} = iJ_2.
\end{aligned}$$

(b) At right are given the weight diagram of two irreps of SO(5) in the basis of T_{12} and T_{34}, *i.e.*, the eigenvalues of these two generators are plotted.. The tick marks are at one unit. For each of these representations (which I'll call the "4" and the "5"), work out what irreps of SU(2) these break into.





To figure this out, we need to figure out the eigenvalues of J_3, which should always be integer or half integer. For the four dimensional irrep, it's pretty clear that these are $-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$. The heighest weight is $\frac{1}{2}$, so there must be a $\left(\frac{1}{2}\right)$ representations. Removing the corresponding weights, which are $-\frac{1}{2}, \frac{1}{2}$, leaves us with the weights $-\frac{1}{2}, \frac{1}{2}$ again, which is simply another $\left(\frac{1}{2}\right)$, so $4 \to \left(\frac{1}{2}\right) \oplus \left(\frac{1}{2}\right)$.

For the 5, we see that the weights are $-1, 0, 0, 0, 1$. The highest weight is 1, so there must be a (1) representation, which should have weights of $-1, 0, 1$. This leaves the weights $0, 0$, which clearly correspond to two copies of the (0) representation, so $5 \to (1) \oplus (0) \oplus (0)$. In conclusion, we have

$$4 \to \left(\tfrac{1}{2}\right) \oplus \left(\tfrac{1}{2}\right) \quad \text{and} \quad 5 \to (1) \oplus (0) \oplus (0).$$

3. An atom in the state $|njm\rangle$ with *j* = 1 is about to decay via electric dipole radiation to the state $|n'j'm'\rangle$. As argued in homework set 25, the probability of it going from an initial state *I* to a final state *F* is proportional to

$$\Gamma(I \to F) = \sum_{q=-1}^{1} \left| \langle F | r_q^{(1)} | I \rangle \right|^2$$

where $r_q^{(1)}$ is the spherical tensor operator corresponding to the vector operator r.

(a) What are the possible final values of *j'*?

Since you are combining a *j* = 1 state with a *k* = 1 tensor operator, the values of *j'* will run from $|j-k|$ to $|j+k|$, so *j'* = 0, 1, or 2.

(b) In fact, it is going to decay to a state with $j' = 1$. Using the Wigner Eckart Theorem, find the *relative* rate of decay

$$\Gamma(njm \to n'j'm')$$

for all non-vanishing possible values of *m* and *m'*.

According to the Wigner-Eckart theorem, the matrix element is proportional to the Clebsch-Gordan coefficient

$$\langle n'j'm' | r_q^{(1)} | njm \rangle \propto \langle j1; qm | j'm' \rangle = \langle 11; qm | 1m' \rangle$$

It will be non-vanishing only if $m' = q + m$, which means we don't really have that many components to calculate. We find, with the help of the online Clebsch Maple routine, that

$$\langle 11;10|11\rangle = -\langle 11;01|11\rangle = \tfrac{1}{\sqrt{2}}$$
$$\langle 11;1,-1|10\rangle = -\langle 11;-1,1|10\rangle = \tfrac{1}{\sqrt{2}}, \quad \langle 11;00|10\rangle = 0,$$
$$-\langle 11;-1,0|1,-1\rangle = \langle 11;0,-1|1,-1\rangle = \tfrac{1}{\sqrt{2}}$$

$m =$	-1	0	1
$m' = -1$	$\frac{1}{2}$	$\frac{1}{2}$	0
$m' = 0$	$\frac{1}{2}$	0	$\frac{1}{2}$
$m' = 1$	0	$\frac{1}{2}$	$\frac{1}{2}$

We then square these to get the relative decay rates, as given in the table above. Interestingly, all the allowed rates are equal, except for the 0 → 0 transition, which has zero probability.

4. **Although the 1, 8, and 10 irreps of Gell-Mann SU(3) are all that are used when we look at combinations of up, down, and strange quarks, other possibilities occur with heavier quarks. For example, the six lightest spin 3/2 baryons containing a charm quark are listed with their mass in the table at right. These particles fit into the 6 irrep of SU(3), which can be written in the form (note: the *c* is not an index, it represents the presence of a charm quark) $\left|B_c^*\right\rangle = u^{ij}\left|B_{c,ij}^*\right\rangle$ with the assignments:**

Name	Mass	I	I_3
Σ_c^{*++}	2518	1	+1
Σ_c^{*+}	2517	1	0
Σ_c^{*0}	2518	1	-1
Ξ_c^{*+}	2647	½	+½
Ξ_c^{*0}	2646	½	-½
Ω_c^{*0}	????	0	0

All masses in MeV/c^2

$$\Sigma_c^{*0}:u^{22}=1,\quad \Sigma_c^{*+}:u^{21}=u^{12}=\tfrac{1}{\sqrt{2}},\quad \Sigma_c^{*++}:u^{11}=1,$$
$$\Xi_c^{*0}:u^{32}=u^{23}=\tfrac{1}{\sqrt{2}},\quad \Xi_c^{*+}:u^{31}=u^{13}=\tfrac{1}{\sqrt{2}},\quad \Omega_c^{*0}:u^{33}=1.$$

(a) **Work out a formula for the mass of these objects in terms of the *u*'s. Include not only an SU(3) respecting piece, but also include a piece where the symmetry is broken proportional to T_8. Include unknown parameters as needed.**

The general form for the mass will take the form

$$m_{B_c^*}=\left\langle B_c^*\right|H\left|B_c^*\right\rangle = au_{ij}^{\dagger}u^{kl}+bu_{ij}^{\dagger}u^{kl}\left(T_8\right)_m^n$$

Now we need to connect up the indices. We have to put all the up indices with down indices. In the first term, because of the symmetry of *u*, it doesn't matter how we match them. In the second term, we have to put one up index and one down index of T_8 each with *u* and $u^\dagger$. It doesn't matter which with which, because of symmetry. The remaining indices then go together. So we have

$$m_{B_c^*}=\left\langle B_c^*\right|H\left|B_c^*\right\rangle = au_{ij}^{\dagger}u^{ij}+bu_{ij}^{\dagger}u^{kj}\left(T_8\right)_k^i$$

(b) **Write an expression for the mass of one of the Σ_c^*'s, one of the Ξ_c^*'s, and the Ω_c^{*0} in terms of the parameters you chose in part (a). Find a linear relationship between them. Predict, on the basis of your relationship, the mass of the Ω_c^{*0}.**

This is straightforward, at least to start with. I'll always do the neutral one

$$m_{\Sigma_c^{*0}}=au_{22}^{\dagger}u^{22}+bu_{22}^{\dagger}u^{22}\left(T_8\right)_2^2=a+\tfrac{1}{2\sqrt{3}}b,$$
$$m_{\Xi_c^{*0}}=au_{23}^{\dagger}u^{23}+au_{32}^{\dagger}u^{32}+bu_{23}^{\dagger}u^{23}\left(T_8\right)_2^2+bu_{32}^{\dagger}u^{32}\left(T_8\right)_3^3=a\left(\tfrac{1}{2}+\tfrac{1}{2}\right)+b\left(\tfrac{1}{2}\cdot\tfrac{1}{2\sqrt{3}}-\tfrac{1}{2}\cdot\tfrac{1}{\sqrt{3}}\right)$$
$$=a-\tfrac{1}{4\sqrt{3}}b,$$
$$m_{\Omega_c^{*0}}=au_{33}^{\dagger}u^{33}+bu_{33}^{\dagger}u^{33}\left(T_8\right)_3^3=a-\tfrac{1}{\sqrt{3}}b,$$

It is then easy to show that the sum of the two extremes is equal to twice the one in the middle:

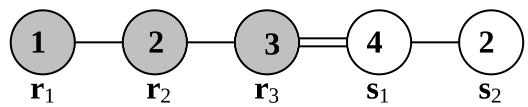
$$m_{\Sigma_c^{*0}} + m_{\Omega_c^{*0}} = 2a - \tfrac{1}{2\sqrt{3}} b = 2m_{\Xi_c^{*0}}$$
$$m_{\Omega_c^{*0}} = 2m_{\Xi_c^{*0}} - m_{\Sigma_c^{*0}} = 2\left(2646.5\right) - 2518 = 2775 \text{ MeV}/c^2$$

The actual mass is around 2768 MeV/c^2; however, all of the masses have measurement errors of a couple of MeV or so, so we can't actually calculate how much we missed by.

Note: This last problem is more like a homework problem; if you are having difficulty with it, come see me, and we will get you unstuck.

5. The last day of class, I presented very abbreviated proofs that certain Dynkin diagrams are not allowed. You are going to elaborate one of them. The diagram at right is an illegal Dynkin diagram, as you will demonstrate.

(a) Define one of the roots to have length *r*. Write the length of each of the five simple roots in terms of *r*.



The three roots $\mathbf{r}_i$ are all connected by single lines, and hence must have the same length. The **s** roots are all shorter by a factor of $\sqrt{2}$. Hence we have

$$\mathbf{r}_1^2 = \mathbf{r}_2^2 = \mathbf{r}_3^2 = r^2, \quad \mathbf{s}_1^2 = \mathbf{s}_2^2 = \tfrac{1}{2} r^2$$

(b) For every pair of simple roots for which the dot product doesn't vanish, write the dot product between them in terms of *r*.

Only roots that are connected have non-vanishing dot-products. We have

$$2\mathbf{r}_1 \cdot \mathbf{r}_2 = \frac{2\mathbf{r}_1 \cdot \mathbf{r}_2}{\mathbf{r}_1^2} \mathbf{r}_1^2 = -r^2, \quad 2\mathbf{r}_2 \cdot \mathbf{r}_3 = \frac{2\mathbf{r}_2 \cdot \mathbf{r}_3}{\mathbf{r}_2^2} \mathbf{r}_2^2 = -r^2,$$
$$2\mathbf{r}_3 \cdot \mathbf{s}_1 = \frac{2\mathbf{r}_3 \cdot \mathbf{s}_1}{\mathbf{r}_3^2} \mathbf{r}_3^2 = -r^2, \quad 2\mathbf{s}_1 \cdot \mathbf{s}_2 = \frac{2\mathbf{s}_1 \cdot \mathbf{s}_2}{\mathbf{s}_1^2} \mathbf{s}_1^2 = -\tfrac{1}{2} r^2.$$

(c) Show that an appropriate combination of the roots above vanishes; *i.e.*, that the square of the combination is zero, and hence this diagram is illegal.

We use the combination given by the numbers, so we have

$$\begin{aligned}\mathbf{v}^2 &= \left(\mathbf{r}_1 + 2\mathbf{r}_2 + 3\mathbf{r}_3 + 4\mathbf{s}_1 + 2\mathbf{s}_2\right)^2 \\ &= \mathbf{r}_1^2 + 4\mathbf{r}_2^2 + 9\mathbf{r}_3^2 + 16\mathbf{s}_1^2 + 4\mathbf{s}_2^2 + 4\mathbf{r}_1 \cdot \mathbf{r}_2 + 12\mathbf{r}_2 \cdot \mathbf{r}_3 + 24\mathbf{r}_3 \cdot \mathbf{s}_1 + 16\mathbf{s}_1 \cdot \mathbf{s}_2 \\ &= r^2 \left(1 + 4 + 9 + 16 \cdot \tfrac{1}{2} + 4 \cdot \tfrac{1}{2} - 2 - 6 - 12 - 8 \cdot \tfrac{1}{2}\right) = 0.\end{aligned}$$