

Physics 215 – Elementary Modern Physics
Complex Numbers

Complex numbers consist of a pair of real numbers, called the real part and the imaginary part, and normally written like this:

$$z = x + iy$$

where x is called the real part and y the imaginary part, which may be written as

$$\operatorname{Re}(z) = x \quad \text{and} \quad \operatorname{Im}(z) = y$$

and i is the square root of minus 1, $i = \sqrt{-1}$.

Adding, subtracting, and multiplying complex numbers is easy, because they satisfy the same properties as real numbers, like the commutative, associative, and distributive properties. Just remember that $i^2 = -1$. For example,

$$(2 + 3i)(4 + 5i) = 8 + 12i + 10i + 15i^2 = 8 - 15 + 22i = -7 + 22i$$

Dividing complex numbers is a bit trickier, but fortunately we don't do it very often. Let's define the *complex conjugate* z^* of a complex number as the same number with the imaginary part reversed, namely,

$$z^* = (x + iy)^* = x - iy$$

Then to divide by a complex number, you can simply multiply the numerator and the denominator by the complex conjugate of the denominator, like this:

$$\frac{2 + 3i}{4 + 5i} = \frac{(2 + 3i)(4 - 5i)}{(4 + 5i)(4 - 5i)} = \frac{8 + 12i - 10i - 15i^2}{16 - 25i^2} = \frac{23 + 2i}{41} = \frac{23}{41} + \frac{2}{41}i$$

The *magnitude* of a complex number is the length of its vector in the complex plane

$$|z| = |x + iy| = \sqrt{x^2 + y^2}$$

A more useful version of this is

$$zz^* = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2 = |z|^2$$

In quantum mechanics, the current status of an electron or other particles is defined by a complex wave function $\psi(\vec{r}, t)$, and the probability density of finding the particle in any given place at any given time is calculated by $|\psi(\vec{r}, t)|^2$.

One very useful formula is the equation

$$e^{i\theta} = \cos \theta + i \sin \theta$$

This formula only works in radians. If you multiply this by its complex conjugate, it is easy to see that you will get 1, so this has magnitude $|e^{i\theta}| = 1$. Related to this expression are two other useful expressions, namely,

$$\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad \text{and} \quad \sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

This expression also allows us to rewrite complex numbers in what we might term “polar” coordinates. Suppose we had a complex number $z = x + iy$. Think of (x,y) as two coordinates in a two-dimensional plane. It is possible to find a distance r and an angle θ such that

$$\begin{aligned} x &= r \cos \theta & \text{namely} & & r &= \sqrt{x^2 + y^2} = |z| \\ y &= r \sin \theta & & & \theta &= \arctan(y/x) \end{aligned}$$

Then the complex number z is

$$z = x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

The quantity r is called the magnitude of z and θ is the phase of z .

Ordinary plane waves can be written in terms of the real and imaginary parts of an exponential like this:

$$\cos(kx - \omega t) = \text{Re} \left[e^{i(kx - \omega t)} \right] \quad \text{and} \quad \sin(kx - \omega t) = \text{Im} \left[e^{i(kx - \omega t)} \right]$$

Such equations allow us to work with the mathematical expression $e^{i(kx - \omega t)}$, which is sometimes simpler than working with sines and cosines, even when we are working with real waves like electromagnetic waves. However, in quantum mechanics, the wave function is actually complex, so that the form $e^{i(kx - \omega t)}$ for a simple wave is, in fact, ideal.