Name ___________________

Solutions to Test 1
September 19, 2018

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered.

Possibly useful formulas:

$\vec{F} = q\vec{E} + q\vec{u}\times\vec{B}$ $p = qRB$	$x' = \gamma(x - vt)$ $t' = \gamma(t - vx/c^2)$ $y' = y,$ $z' = z$	$E' = \gamma(E - vp_x)$ $p'_x = \gamma(p_x - vE/c^2)$ $p'_y = p_y, \quad p'_z = p_z$	$u'_x = \dfrac{u_x - v}{1 - vu_x/c^2}$ $u'_y = \dfrac{u_y}{\gamma(1 - vu_x/c^2)}$ $u'_z = \dfrac{u_z}{\gamma(1 - vu_x/c^2)}$
$\dfrac{\vec{u}}{c} = \dfrac{\vec{p}c}{E}$		$e = 1.602\times10^{-19}$ C	
$f = \dfrac{f_0}{\gamma(1 - v\cos\theta/c)}$	$(1+\varepsilon)^n = 1 + n\varepsilon + \frac{1}{2}n(n-1)\varepsilon^2 + \cdots$		

Part I: Multiple Choice [20 points]
For each question, choose the best answer (2 points each)

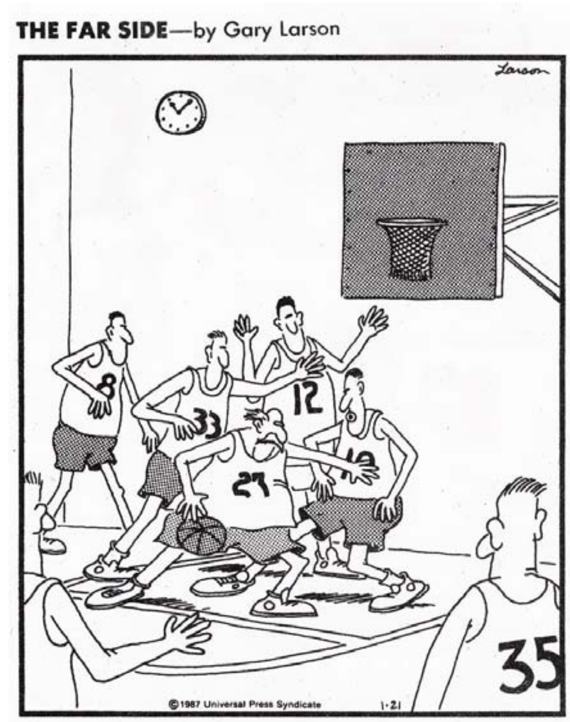
1. According to special relativity, the speed of light in vacuum depends on
   A) The speed of the object emitting it (only)
   B) The speed of the observer (only)
   C) Both the speed of the observer and of the emitter
   D) Neither the speed of the observer nor the emitter
   E) I have no idea; please mark this one wrong

2. What, if anything, is wrong with the formula $s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 + (\Delta t)^2$?
   A) The space components should all have the opposite sign (only)
   B) The time component should have the opposite sign (only)
   C) The time component needs to be multiplied by c^2 (only)
   D) Both B and C are true
   E) Nothing; the formula is fine as is

3. When we first tried to "check" conservation of momentum, we found that it seemed to work in one frame but not the other. The reason this happened is because
   A) Conservation of momentum is only true in the center of mass coordinate system
   B) Conservation of momentum is generally not true in relativity
   C) The formula for momentum had to be changed in special relativity
   D) The process we considered was in fact impossible in relativity
   E) We falsely assumed that there are rigid objects in relativity

4. How fast does a clock advance according to an observer moving compared to the clock?
   A) The clock runs slower than normal
   B) The clock runs faster than normal
   C) The clock runs at the same speed as normal
   D) The clock runs slower only if the clock is truly stationary and the observer is moving
   E) The clock runs slower only if the clock is truly moving and the observer is stationary

5. In certain stars, three ^{4}He nuclei are fused to make a ^{12}C nucleus, and in the process energy is released. How does the mass of one ^{12}C nucleus compare to the mass of three ^{4}He nuclei?
   A) The one ^{12}C nucleus has more mass
   B) The three ^{4}He nuclei have more mass
   C) They have exactly the same mass
   D) The question can only be answered if we know the amount of energy released
   E) The question cannot be answered, even if we knew the amount of energy released



Unbeknownst to most historians, Einstein started down the road of professional basketball before an ankle injury diverted him into science.

6. A proton has a mass of 0.938 GeV/c^2. If it had a total energy of 1000 GeV, what would be its approximate speed?
   A) 0.000469*c* B) 0.0217*c* **C) *c*** D) 46.2*c* E) 2132*c*

7. Which of the following is <u>not</u> a prediction of relativity?
   A) The maximum momentum a particle of mass *m* can achieve is *mc*
   B) No object can move faster than light
   C) There is no such thing as a rigid object
   D) When you add internal energy to an object, its mass increases
   E) Whether two events are "simultaneous" depends on the motion of the observer

8. If the <u>kinetic</u> energy of a particle were mc^2, what would be the value of the Lorentz factor γ?
   A) -1 B) 0 C) 1 D) $\sqrt{2}$ **E) 2**

9. The separation between your professor's birth and here and now is
   A) Spacelike **B) Timelike** C) Lightlike D) None of these E) Insufficient information

10. In special relativity, momentum becomes a four-dimensional vector, with the fourth component corresponding to
   A) Time B) Space C) Speed D) Mass **E) Energy**

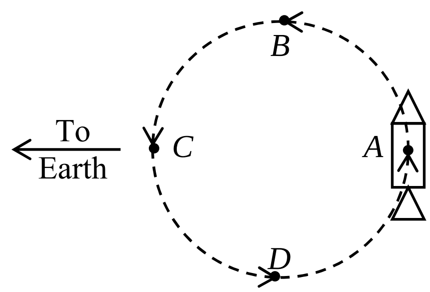
Part II: Short answer [20 points]

Choose **two** of the following questions and give a short answer (1-3 sentences) (10 points each).

11. A ribbon is moving at high speed when two scissors simultaneously cut the ribbon at two places exactly one meter apart. The cut ribbon is then brought to rest, and found to be more than a meter long. How can you explain this? According to an observer moving with the scissors, the scissors were less than a meter apart. How then can the cut piece be longer than a meter?

The length of ribbon being cut has a proper length that is greater than its length as measured in the rest frame of the scissors, due to Lorentz contraction. Although the scissor cut the ribbon in their own frame simultaneously, in the frame of the ribbon, they were not cut simultaneously, and hence the cuts would come at different times. From the ribbon's perspective, the front one was cut first, then the ribbon kept coming through, and the back one was cut later.

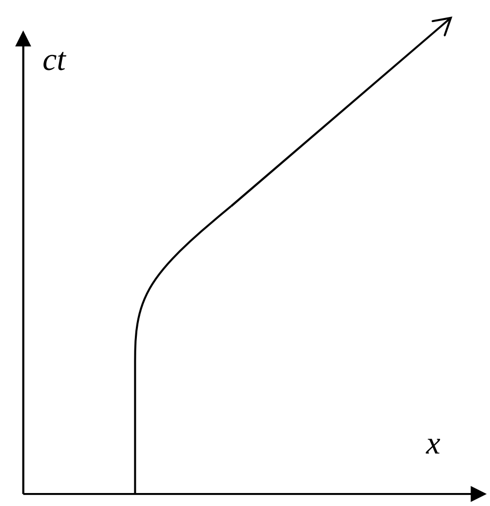
12. A spaceship is flying in a counter-clockwise circle far from us at high velocity. It is transmitting signals at a frequency f_0. At each of the four points marked, will the received signal at Earth be higher, lower, or the same as f_0? At which of the points will it be highest and lowest?



The frequency will be higher (blue shifted) at point B, lower (red shifted) at point D, and also lower ("orange shifted") at points C and A. It will be highest at B (of course), when it is moving directly towards the observer, but lowest at D when it is moving directly away from the observer.

13. Make a crude sketch of the world line of a particle initially at rest which then accelerates in the +*x* direction for a short time, then travels at constant velocity.

The sketch appears at right



Part III: Calculation: [60 points]
Choose **three** of the following four questions and perform the indicated calculations (20 points each)

14. A collection of 1000 atoms of ^{294}Og atoms are produced at high but uniform speed. It is found that after traveling a distance of 191 km in a time of 1.13 ms**, 302 of them still exist (the rest decayed).**

(a) What is the speed of these particles?

Velocity is distance over time, so

$$v = \frac{d}{\Delta t} = \frac{1.91\times10^5\ \text{m}}{1.13\times10^{-3}\ \text{s}} = 1.69\times10^8\ \text{m/s}.$$

(b) According to an observer moving with these particles, how long do they last?

According to the time-dilation formula, $\Delta t = \gamma\tau$, so we have

$$\tau = \frac{\Delta t}{\gamma} = \Delta t\sqrt{1-\frac{v^2}{c^2}} = (1.13\ \text{ms})\sqrt{1-\left(\frac{1.69\times10^8\ \text{m/s}}{2.998\times10^8\ \text{m/s}}\right)^2} = (1.13\ \text{ms})\sqrt{1-0.568^2} = 0.933\ \text{ms}.$$

(c) The probability of a particle not decaying after a time τ as viewed in its own frame is given by $P = e^{-\tau/\tau_0}$, where τ_0 is the mean lifetime. What is the mean lifetime of ^{294}Og?

Obviously, if 302 out of 1000 survie, then the probability of surviving this long is $P = 0.302$. Taking the natural log of both sides, we have

$$\ln(P) = -\frac{\tau}{\tau_0},$$

$$\tau_0 = \frac{-\tau}{\ln(P)} = \frac{-0.933\ \text{ms}}{\ln(0.302)} = \frac{-0.933\ \text{ms}}{-1.197} = 0.779\ \text{ms}.$$

15. An American holding a yardstick (1 yard = 0.9144 m) is moving at high speed compared to a European with a meter stick. The American notes that the meter stick is, according to him, 0.9144 yards long (he is moving in the direction of the length of the yard stick).

(a) How fast is the American moving?

The meter stick has an apparent length of

$$L = 0.9144 \text{ yard} = 0.9144\left(0.9144 \text{ m}\right) = 0.8361 \text{ m}.$$

This is shorter than the proper length, which is presumably $L_p = 1 \text{ m}$ due to Lorentz contraction. Rearranging the formula $L = L_p/\gamma$, we have

$$\frac{1}{\gamma} = \frac{L}{L_p} = \frac{0.8361 \text{ m}}{1.000 \text{ m}} = 0.8361 = \sqrt{1 - \frac{v^2}{c^2}},$$

$$1 - \frac{v^2}{c^2} = 0.8361^2 = 0.6991,$$

$$\frac{v^2}{c^2} = 1 - 0.6991 = 0.3009,$$

$$v = c\sqrt{0.3009} = 0.5485\left(2.998 \times 10^8 \text{ m/s}\right) = 1.645 \times 10^8 \text{ m/s}.$$

(b) How long does the yard stick look, according to the European?

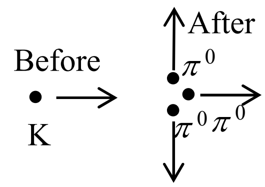
The yard stick has an actual length of L_p = 0.9144 m, but it is reduced to $L = L_p/\gamma$, so

$$L = \frac{L_p}{\gamma} = \left(0.9144 \text{ m}\right)\left(0.8361\right) = 0.7646 \text{ m}.$$

(c) The yard stick is 2.00 in across (1 in = 2.54 cm). How wide is the yard stick, according to the European?

Lorentz contraction occurs only in the direction of motion, not perpendicular to it, so the width is 2.00 in = 5.080 cm.

16. A K-meson moving to the right decays to three neutral pions (m_π = 135.0 MeV/c^2), one moving to the right, one moving straight down, and one moving straight up. All three of the neutral pions have speed 0.604*c*, though in different directions.



(a) What is the energy of each of the three pions in MeV?

The energy of the pions is given by $E = \gamma mc^2$, so we have

$$E_\pi = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \frac{\left(135.0 \text{ MeV}/c^2\right)c^2}{\sqrt{1 - 0.604^2}} = 169.4 \text{ MeV}.$$

(b) What is the momentum (magnitude and direction) of each of the three pions in MeV/*c*?

The magnitude of the momentum is, for each of them, given by $p = \gamma mv$, so we have

$$p_\pi = \gamma mv = \frac{mv}{\sqrt{1 - v^2/c^2}} = \frac{\left(135.0 \text{ MeV}/c^2\right)\left(0.604c\right)}{\sqrt{1 - 0.604^2}} = 102.3 \text{ MeV}/c.$$

However, momentum is a vector, and each of these have a direction. If we define right as the positive *x*-direction, and up as the positive *y*-direction, then these momenta are

$$\vec{p}_{1\pi} = 102.3\hat{\mathbf{i}} \text{ MeV}/c, \quad \vec{p}_{2\pi} = 102.3\hat{\mathbf{j}} \text{ MeV}/c, \quad \vec{p}_{3\pi} = -102.3\hat{\mathbf{j}} \text{ MeV}/c.$$

(c) What is the energy (in MeV) and momentum (in MeV/*c*) of the initial K-meson?

By conservation of energy and momentum, this must just be the sum of the final energies and momenta. Note that $\vec{p}_{2\pi}$ and $\vec{p}_{3\pi}$ will cancel. So we have

$$E_K = 3E_\pi = 3\left(169.4 \text{ MeV}\right) = 508.2 \text{ MeV},$$

$$\vec{p}_K = \vec{p}_{1\pi} + \vec{p}_{2\pi} + \vec{p}_{3\pi} = 102.3\left(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{j}}\right)\text{MeV}/c = 102.3\hat{\mathbf{i}} \text{ MeV}/c.$$

(d) What is the mass (in MeV/c^2) of the K-meson?

We use the formula $E^2 - \vec{p}^2c^2 = m^2c^4$ to find

$$m_K^2c^4 = E_K^2 - \vec{p}_K^2c^2 = \left(508.2 \text{ MeV}\right)^2 - \left(102.3 \text{ MeV}\right)^2 = 247800 \text{ MeV}^2,$$

$$m_Kc^2 = \sqrt{247800 \text{ MeV}^2} = 497.8 \text{ MeV},$$

$$m_K = 497.8 \text{ MeV}/c^2.$$

17. A colonization spaceship, initially at rest, is headed to Tau Ceti ($d = 1.126\times10^{17}\ \text{m}$) and has a mass of 5.10 × 10^6 kg and has a constant force of 5.00 × 10^7 N on it.

(a) What is the initial energy of this spaceship?

Since the spaceship is at rest, $E_i = mc^2$, so

$$E_i = mc^2 = \left(5.10\times10^6\ \text{kg}\right)\left(2.998\times10^8\ \text{m/s}\right)^2 = 4.58\times10^{23}\ \text{J}.$$

(b) The force is maintained for <u>half</u> the distance to Tau Ceti. What is the energy of the spaceship at the half-way point?

We have $\Delta E = W = Fd$, so

$$E_f = E_i + W = E_i + F\left(\tfrac{1}{2}d\right) = E_i + \tfrac{1}{2}\left(5.00\times10^7\,\text{N}\right)\left(1.126\times10^{17}\ \text{m}\right)$$
$$= \left(4.58\times10^{23}\ \text{J}\right) + \left(2.81\times10^{24}\ \text{J}\right) = 3.27\times10^{24}\ \text{J}.$$

(c) What is the speed of the ship at the half-way point?

We use the formula $E = \gamma mc^2$, which we rewrite as

$$\frac{1}{\gamma} = \frac{mc^2}{E} = \frac{4.58\times10^{23}\ \text{J}}{3.27\times10^{24}\ \text{J}} = 0.1400,$$
$$1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2} = 0.1400^2 = 0.0196,$$
$$\frac{v^2}{c^2} = 1 - .0196 = 0.9804,$$
$$v = \sqrt{0.9804}c = 0.9901\left(2.998\times10^8\ \text{m/s}\right) = 2.968\times10^8\ \text{m/s}.$$

(d) What is the momentum of the ship at the half-way point?

The momentum is given by

$$p = \gamma mv = \frac{E}{mc^2}mv = \frac{3.27\times10^{24}\ \text{J}}{4.58\times10^{23}\ \text{J}}\left(5.10\times10^6\ \text{kg}\right)\left(2.968\times10^8\ \text{m/s}\right) = 1.081\times10^{16}\ \text{kg}\cdot\text{m/s}.$$

(e) How long (in years) does it take to reach this point ($\text{y} = 3.156\times10^7\ \text{s}$)?

We comput this using $\Delta p = F\cdot\Delta t$. The starting momentum was zero, so we have

$$\Delta t = \frac{\Delta p}{F} = \frac{1.081\times10^{16}\ \text{kg}\cdot\text{m/s}}{5.00\times10^7\ \text{kg}\cdot\text{m/s}^2} = \left(2.162\times10^8\ \text{s}\right)\frac{1\,\text{y}}{3.156\times10^7\ \text{s}} = 6.85\ \text{y}.$$