Solutions to Test 1
September 18, 2019

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered.

Possibly useful formulas:

$x' = \gamma(x - vt)$ $t' = \gamma(t - vx/c^2)$ $y' = y, \quad z' = z$	$E' = \gamma(E - vp_x)$ $p'_x = \gamma(p_x - vE/c^2)$ $p'_y = p_y, \quad p'_z = p_z$	$f = \dfrac{f_0}{\gamma(1 - v\cos\theta/c)}$ $e = 1.602 \times 10^{-19}$ C	$u'_x = \dfrac{u_x - v}{1 - vu_x/c^2}$ $u'_y = \dfrac{u_y}{\gamma(1 - vu_x/c^2)}$ $u'_z = \dfrac{u_z}{\gamma(1 - vu_x/c^2)}$
$\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}$ $p = qRB$	$(1+\varepsilon)^n = 1 + n\varepsilon + \frac{1}{2}n(n-1)\varepsilon^2 + \cdots$	$\dfrac{\vec{u}}{c} = \dfrac{\vec{p}c}{E}$	

Part I: Multiple Choice [20 points]
For each question, choose the best answer (2 points each)

1. Two observers are moving relative to each other, and each tries to measure the other's meter sticks, which are oriented in the direction of their relative motion. What will they conclude about each other's meter sticks?
   A) They will both agree that the one who is truly at rest will have the shorter meter stick
   B) They will both agree that the one who is truly moving will have the shorter meter stick
   C) Each one will see the other's stick as longer than their own
   D) Each one will see the other's stick as shorter than their own
   E) Insufficient information

2. When is the formula $E = mc^2$ valid?
   A) Always
   B) Only when the object is at rest
   C) Only when the object is moving
   D) Only on Friday the 13th
   E) Never

3. The total energy of an object of mass *m* and velocity *v* is
   A) $\dfrac{mc^2}{\sqrt{1 - v^2/c^2}}$ B) $mc^2\sqrt{1 - v^2/c^2}$ C) mc^2 D) $\frac{1}{2}mv^2$ E) $\dfrac{mvc}{\sqrt{1 - v^2/c^2}}$

4. If you pushed on an object initially at rest with mass *m* with a force *F*, how long would you have to push to get the object to the speed of light *c*?
   A) $t = \dfrac{F}{mc}$ B) $t = \dfrac{Fm}{c}$ C) $t = \dfrac{Fc}{m}$ D) $t = \dfrac{mc}{F}$ **E) Forever; you can never reach *c***

5. If events P_1 and P_2 have a timelike separation, what could be the relationship of P_2 to P_1?
   A) P_2 is in the absolute future of P_1 (only)
   B) P_2 is in the absolute past of P_1 (only)
   C) P_2 is elsewhere compared to P_1 (only)
   D) Both A and B are possible, but not C
   E) A, B, and C are all possibilities

6. It is possible to collide a proton and anti-proton (each mass m_p) to make three neutral pions (each mass m_π). From this information alone, what can we conclude about the pion mass?
   A) $m_\pi = 2m_p$ B) $m_\pi = \frac{2}{3}m_p$ C) $m_\pi = \frac{3}{2}m_p$ D) $m_\pi = 3m_p$ **E) Nothing**

7. If you tried to communicate using a perfectly rigid rod by pushing on one end while measuring the position of the other, you would find
   A) The other end moves simultaneously as viewed in all reference frames
   B) The other end moves simultaneously only in the frame where the rod is not moving
   C) The other end moves simultaneously only in the frame where the signaler is not moving
   D) The other end moves simultaneously only in the frame where the receiver is not moving
   E) Rigid rods are impossible in special relativity, so the question makes no sense

8. If you have two events (points in spacetime), which of the following will all observers agree on?
   A) Their separation in space (only)
   B) The difference in time between them (only)
   C) The proper distance squared, s^2 (only)
   D) The separation in space and in time, but not the proper distance squared
   E) The separation in space and in time as well as the proper distance

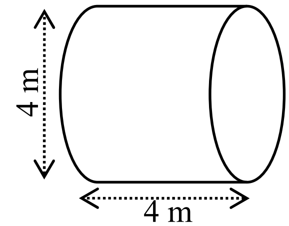
9. An electron has a mass squared that is _________ and a photon has a mass squared that is _______.
   A) Positive, positive
   B) Positive, zero
   C) Positive, negative
   D) Zero, positive
   E) Zero, zero

10. A particle of mass m initially pushed by a force F will experience a constant acceleration $a = F/m$, causing it to reach a speed $v = at$ and travel a distance $d = \frac{1}{2}at^2$, and reaching the speed of light after a time $t = c/a$ in a distance $d = c^2/2a$. Which of these formulas is the first one where I made a mistake, according to relativity?
    A) $a = F/m$ B) $v = at$ C) $d = \frac{1}{2}at^2$ D) $t = c/a$ E) $d = c^2/2a$

Part II: Short answer [20 points]

Choose **two** of the following questions and give a short answer (1-3 sentences) (10 points each).

11. A cylindrical spaceship is 4.00 m in length and 4.00 m in diameter. It is traveling so fast that it has a Lorentz factor of $\gamma = 2.00$. Describe its apparent shape, as observed by us, if it is moving (a) to the right ➔ at this speed or (b) moving up 🡅 at this speed.



The cylinder gets Lorentz contracted by a factor of $\gamma = 2.00$, but only in the direction of motion. If it is moving to the right, it will still be a cylinder, but it will only be 2 m in length, while retaining its round profile with the same diameter. If it is moving upwards, it will get squashed in this direction, so it will still be 4 m left-right, but it will be squashed in the vertical direction by a factor of 2, and hence will be an ellipse, only 2 m up and down while still being 4 m out of the plane of the paper.

12. The Tevatron was an approximately circular particle collider 1.00 km in radius that could accelerate protons up to a momentum of 0.980 TeV/*c*. The LHC uses slightly better magnets, is 4.30 km in radius, and can get the protons up to a momentum of 7.00 TeV/*c*. Explain what magnets have to do with it, and explain what advantage the LHC has that allows it to reach higher momenta. You should include at least one equation in your answer.

The particles must be kept in a circle by the presence of a magnetic field, as they are getting pushed sideways by the force $\vec{F} = q\vec{u} \times \vec{\mathrm{B}}$. The momentum is then governed by $p = qRB$. You can't adjust the charge, but the slightly improved magnetic field *B* helps you reach higher momentum, but more important is the much larger radius *R*.

13. In a homework problem, you found that a "bunch" of protons, when it passes through the Atlas detector, as viewed by us is much shorter than the Atlas detector, but if you were moving along with them it is much longer than the Atlas detector. But surely the protons are either all in the detector at the same time or not. Resolve this apparent paradox.

The question asks "at the same time," but of course, in relativity, the definition of simultaneity is ambiguous. This is basically identical with the barn-pole paradox from the book, where measuring the two ends of the bunch (or the barn) will not be simultaneous in one frame when it is in the other frame.

Part III: Calculation: [60 points]
Choose **three** of the following four questions and perform the indicated calculations (20 points each)

14. A collection of 4.17×10^6 **muons is traveling around a circle of <u>radius</u> *r* = 26.9 m, completing a circuit every 0.710 μs.**

(a) What is the speed of these particles?

They are going in a circle, which means they are going a distance of $C = 2\pi r$ in one cycle. Speed is distance over time, so we have

$$v = \frac{C}{\Delta t} = \frac{2\pi(26.9\text{ m})}{0.710\times10^{-6}\text{ s}} = \frac{169.1\text{ m}}{0.710\times10^{-6}\text{ s}} = 2.38\times10^8\text{ m/s}.$$

(b) From the viewpoint of the muons, how long does it take them to go around once?

We know the period is $\Delta t = 0.710\ \mu\text{s}$, and we use the formula $\Delta t = \gamma\tau$ and solve for τ:

$$\tau = \frac{\Delta t}{\gamma} = \Delta t\sqrt{1-\frac{v^2}{c^2}} = (0.710\ \mu\text{s})\sqrt{1-\left(\frac{2.38\times10^8\text{ m/s}}{2.998\times10^8\text{ m/s}}\right)^2} = (0.710\ \mu\text{s})\sqrt{1-0.794^2}$$
$$= (0.710\ \mu\text{s})\sqrt{0.3695} = 0.4315\ \mu\text{s}.$$

(c) The probability of a muon not decaying after a time τ **as viewed in its own frame is given by** $P = e^{-\tau/\tau_0}$**, where** $\tau_0 = 2.197\ \mu\text{s}$ **is the mean lifetime. How many muons are left after <u>ten</u> complete circuits?**

Ten circuits takes en times as long as one circuit, or $\tau = 10(0.4316\ \mu\text{s}) = 4.316\ \mu\text{s}$. We therefore have

$$P = e^{-\tau/\tau_0} = \exp\left(-\frac{4.316\ \mu\text{s}}{2.197\ \mu\text{s}}\right) = e^{-1.964} = 0.1402.$$

We therefore simply multiply this by the starting number to find

$$N_f = PN_i = 0.1402\left(4.17\times10^6\right) = 5.85\times10^5.$$

15. A beam of light start at the origin, $(x,y,z,t)=(0,0,0,0)$, traveling in the +*y* direction. It is detected by two detectors each 2.00 m away from the initial event.

(a) What are all four coordinates (x,y,z,t) when the light beam is detected?

Since it's traveling in the +*y* direction, we will still have $x = z = 0$. The time can be found from $d = ct$, so we have

$$t=\frac{d}{c}=\frac{2.00\text{ m}}{2.998\times 10^{8}\text{ m/s}}=6.67\times 10^{-9}\text{ s}=6.67\text{ ns}.$$

So we have $(x,y,z,t)=(0,2.00\text{ m},0,6.67\text{ ns})$.

(b) An observer traveling at $v = 1.80\times 10^8$ m/s in the +*x* direction observes the emission and detection. Find the coordinates of the detection event (and emission event) (x',y',z',t') as viewed by this observer.

It is trivial to work out that the emission event is at $(x',y',z',t')=(0,0,0,0)$. For the detection event, we have

$$x'=\gamma(x-vt)=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{0-(1.80\times 10^{8}\text{ m/s})(6.67\times 10^{-9}\text{ s})}{\sqrt{1-\left(\frac{1.80\times 10^{8}\text{ m/s}}{2.998\times 10^{8}\text{ m/s}}\right)^2}}=\frac{-1.201\text{ m}}{\sqrt{1-0.600^2}}=-1.50\text{ m},$$

$$t'=\gamma\left(t-vx/c^2\right)=\frac{t-vx/c^2}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{(6.67\text{ ns})-0}{\sqrt{1-\left(\frac{1.80\times 10^{8}\text{ m/s}}{2.998\times 10^{8}\text{ m/s}}\right)^2}}=\frac{6.67\text{ ns}}{\sqrt{1-0.600^2}}=8.34\text{ ns},$$

$$y'=y=2.00\text{ m},\quad z'=z=0.$$

So we have $(x',y',z',t')=(-1.50\text{ m},2.00\text{ m},0,8.34\text{ ns})$.

(c) Calculate the components of the velocity of light $\vec{v}=(v_x,v_y,v_z)$ using the locations and time you found in part (b).

Velocity is just distance over time, so we have

$$v_x=\frac{-1.50\text{ m}}{8.34\times 10^{-9}\text{ s}}=-1.80\times 10^{8}\text{ m/s},\quad v_y=\frac{2.00\text{ m}}{8.34\times 10^{-9}\text{ s}}=2.40\times 10^{8}\text{ m/s},\quad v_z=0.$$

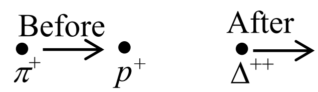
(d) Calculate the speed (magnitude of the velocity) using the numbers you found in part (c). Comment on whether the answer makes sense or not.

The magnitude of the velocity can found by taking the square root of the sum of the components, so we have

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\left(-1.80\times10^8\ \text{m/s}\right)^2 + \left(2.40\times10^8\ \text{m/s}\right)^2 + 0^2} = 3.00\times10^8\ \text{m/s}\,.$$

This is just the speed of light back again, as the speed of light is always the same according to all observers.

16. The Δ^{++} baryons is produced by colliding positively charges pions π^+ (m_π = 139 MeV/c^2) with an energy of E_π = 328 MeV with protons at rest (m_p = 938 MeV/c^2).



(a) What is the momentum (in MeV/*c*) and speed (as a fraction of *c*) of the pions?

The momentum is most easily computed using the formula $E^2 - p^2c^2 = \left(mc^2\right)^2$ to solve for the momentum. We have

$$p^2c^2 = E^2 - \left(mc^2\right)^2 = \left(328\ \text{MeV}\right)^2 - \left(139\ \text{MeV}\right)^2 = 88260\ \text{MeV}^2\,,$$

$$pc = \sqrt{88260\ \text{MeV}^2} = 297\ \text{MeV}\,.$$

So $p = 297\ \text{MeV}/c$. We then can find the speed as a fraction of *c* using

$$\frac{u}{c} = \frac{pc}{E} = \frac{297\ \text{MeV}}{328\ \text{MeV}} = 0.906\,.$$

(b) What is the energy and momentum of the resulting Δ^{++}?

Since the proton is at rest, it has zero momentum and energy $E_p = E_0 = m_pc^2 = 938\ \text{MeV}$. We just use conservation of momentum and energy to conclude that

$$p_\Delta = p_\pi + p_p = \left(297\ \text{MeV}/c\right) + 0 = 297\ \text{MeV}/c\,,$$
$$E_\Delta = E_\pi + E_p = \left(328\ \text{MeV}\right) + \left(938\ \text{MeV}\right) = 1266\ \text{MeV}\,.$$

(c) What is the mass (in MeV/c^2) and speed (as a fraction of *c*) of the resulting Δ^{++}?

We use the same two formulas as before to find

$$\left(m_\Delta c^2\right)^2 = E_\Delta^2 - \left(p_\Delta c\right)^2 = \left(1266\text{ MeV}\right)^2 - \left(297\text{ MeV}\right)^2 = 1,515,000\text{ MeV}^2\,,$$

$$m_\Delta c^2 = \sqrt{1,515,000\text{ MeV}^2} = 1231\text{ MeV}\,,$$

$$\frac{u}{c} = \frac{p_\Delta c}{E_\Delta} = \frac{297\text{ MeV}}{1266\text{ MeV}} = 0.235\,.$$

17. Neutral hydrogen atoms at rest emit radio waves at a frequency of 1420.4 MHz. A cloud of hydrogen atoms is moving directly away from us at $v = 172,000\text{ km/s}$.

(a) At what frequency will we observe the radio waves from this source?

In this case, since the gas is moving directly away from us, we have $\theta = \pi$, and therefore $\cos\theta = -1$. We first find the speed as a fraction of the speed of light, and the Lorentz factor, which are

$$\frac{v}{c} = \frac{1.72\times10^8\text{ m/s}}{2.998\times10^8\text{ m/s}} = 0.574\,,$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-0.574^2}} = \frac{1}{\sqrt{0.671}} = 1.221\,.$$

The detected frequency is then

$$f = \frac{f_0}{\gamma\left(1 - v\cos\theta/c\right)} = \frac{f_0}{\gamma\left(1+v/c\right)} = \frac{1420.4\text{ MHz}}{1.221\left(1+0.574\right)} = 739.1\text{ MHz}\,.$$

(b) If the cloud of gas has a mass $m = 1.989\times10^{30}\text{ kg}$**, what is its momentum?**

The momentum is just given by

$$p = \gamma m v = 1.221\left(1.989\times10^{30}\text{ kg}\right)\left(1.72\times10^8\text{ m/s}\right) = 4.178\times10^{38}\text{ kg}\cdot\text{m/s}\,.$$

(c) If a constant force of $F = 1.56\times10^{26}\text{ N}$ **were applied to slow down this cloud, how long would it take in years to bring it to rest? (1 y = 3.156×10^7 s)**

We start with the formula $F = dp/dt$, which for a constant force turns into $F\Delta t = \Delta p$. We therefore have

$$\Delta t = \frac{\Delta p}{F} = \frac{4.178\times10^{38}\text{ kg}\cdot\text{m/s}}{1.56\times10^{26}\text{ kg}\cdot\text{m/s}^2} = \left(2.67\times10^{12}\text{ s}\right)\cdot\frac{1\text{ y}}{3.156\times10^7\text{ s}} = 84,800\text{ y}\,.$$