Solutions to Test 1
September 18, 2020

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered.

Part I: Multiple Choice [20 points]
For each question, choose the best answer (2 points each)

2. In relativity, vectors should have four components. What is the fourth component of the momentum vector?
   A) Space B) Time **C) Energy** D) Mass E) Proper time

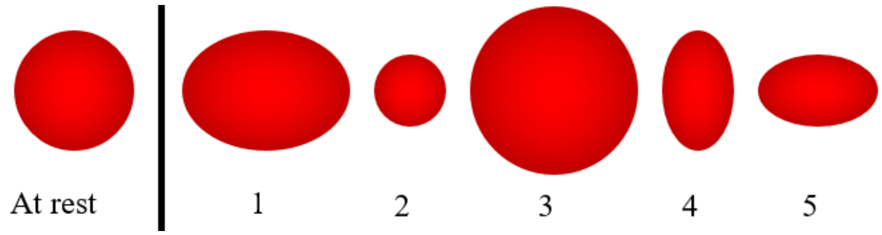
3. Below is a four step proof that $\vec{F} = m\vec{a}$. Which step, if any, is wrong?
   A) $\vec{F} = \frac{d\vec{p}}{dt}$
   B) $\frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$
   C) $\frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt}$
   D) $m\frac{d\vec{v}}{dt} = m\vec{a}$
   E) None of these are wrong

4. Suppose we took an uncompressed spring of mass m and then compressed it. What effect would this have on the mass?
   A) It would decrease a lot
   B) It would decrease a little
   C) It would stay the same
   D) It would increase a little
   E) It would increase a lot

5. As a particle of mass m approaches the speed of light, the momentum approaches ___________ and the kinetic energy approaches _____________ .
   A) $mc, \frac{1}{2}mc^2$ B) mc, mc^2 C) mc, ∞ D) ∞, mc^2 **E)** ∞, ∞

6. Which of the following is **not** a prediction of special relativity?
   A) The speed of light will look different to different observers.
   B) There are no rigid objects
   C) Which of two events occurs first may be disagreed on by two observers
   D) Putting internal energy into an object increases its mass
   E) Moving clocks, according to a stationary observer, run more slowly.
7. Suppose the image below left represents an object at rest. Which of the other images could represent the same objects moving to the right at high velocity?

A) 1 B) 2 C) 3 **D) 4** E) 5



8. In a particle collider, the particles are kept in a circular track by placing _______________ fields on them that are ___________ to the direction of motion.
   A) Magnetic, parallel
   B) Magnetic, perpendicular
   C) Electric, parallel
   D) Electric, perpendicular
   E) Gravitational, parallel

9. The speed of light in vacuum as measured by a moving observer depends on which of the following?
   A) The speed of the source towards or away from you (only)
   B) The speed of the source lateral to you (only)
   C) The speed of the observer (only)
   D) All three of the above
   E) None of the above

10. Suppose two observers both start at event *A* and end at event *B*. Observer 1 moved at constant velocity, while observer 2 accelerated. Which one will have experienced more proper time?
    A) Observer 1
    B) Observer 2
    C) They will experience the same amount of time
    D) Insufficient information
    E) I have no idea; please mark this one wrong

11. According to relativity, if an object moving very close to the speed of light is emitting light, which direction does most of that light go, as viewed in a frame not moving with the object?
    A) About equal in all directions
    B) Backwards (only)
    C) Forwards (only)
    D) Perpendicular to the direction of motion
    E) Forwards and backwards, but not sideways

Part II: Short essay [20 points]

Choose two of the following three questions, and write a short essay (2-3 sentences). You may type both answers into the answer box at the end, or you may upload your answers as an image into the box. Each question is worth 10 points.

12A. Suppose two identical spaceships fly past each other. You can tell which one is actually moving by seeing which of their clocks is running slower. Comment on this statement.

This statement is incorrect. Each of the two observers will claim that the others' clock is moving slowly.

12B. Since which of two events happens first is ambiguous in relativity, is it possible that according to some observers, you finished this test even before you started it? Explain your reasoning.

For events that have *timelike* separation (where there's a lot of time separating them, and little distance), all observers will agree. In particular, everyone will agree that I started this test well before I finished it.

12C. If you have a container filled with gas and you heat it, the mass of each molecule does not change, they just move (on the average) faster. How, then, can the mass of the total change when you heat it? Or does it change?

Any form of energy added to an object (other than bulk kinetic energy) will increase its mass. In particular, if you heat an object, individual molecules will have momentum, but the system as a whole will have no net momentum. The mass of the composite object is not the sum of the masses of its constituents.

Part III: Calculation: [60 points]

Choose three of the following four questions. Each question is worth 20 points. Type only your answers to each part into the essay box provided.

13. A group of 48,000 ^{216}Po atoms are accelerated to high speed and sent in a circular path circling the Earth (radius = 6370 km), such that they circle it in 0.230 s.

a) What is the velocity of these atoms? What is the Lorentz factor γ?

Because they are traveling in a circle, the distance is given by

$$C = 2\pi R = 2\pi\left(6370\times10^3 \text{ m}\right) = 4.002\times10^7 \text{ m}$$

The velocity is just the distance over time, or

$$v = \frac{C}{t} = \frac{4.002\times10^7 \text{ m}}{0.230 \text{ s}} = 1.740\times10^8 \text{ m/s}.$$

Therefore, the Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{1.740\times10^8 \text{ m/s}}{2.998\times10^8 \text{ m/s}}\right)^2}} = 1.228.$$

b) It is found that only 19,600 of them remain after circling the Earth once. This is because they radioactively decay, with the probability of survival given by $P = e^{-\lambda\tau}$, where τ is the proper time and λ is the decay constant. What is λ in units of s^{-1}?

The proper time for orbiting the Earth can be found from $\Delta t = \gamma\tau$, so

$$\tau = \frac{\Delta t}{\gamma} = \frac{0.230 \text{ s}}{1.228} = 0.187 \text{ s}.$$

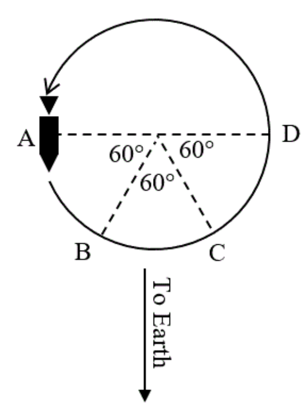
Taking the natural log of both sides of the probability formula, we have $\ln(P) = -\lambda\tau$, which we can solve for $\lambda = -\ln(P)/\tau$. The probability is the fraction that survive, so

$$P = \frac{19,600}{48,000} = 0.4083.$$

Substituting everything in, we therefore have

$$\lambda = -\frac{\ln(P)}{\tau} = -\frac{\ln(.4083)}{0.187 \text{ s}} = \frac{0.896}{0.187 \text{ s}} = 4.79 \text{ s}^{-1}.$$

14. A cylindrical spaceship of length 260. m and diameter 21.0 m is moving in a circular orbit.



a) **What is the Lorentz factor γ? What is the ship's speed? What is the apparent diameter of the ship, according to a stationary observer?**

The Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-0.800^2}} = \frac{1}{\sqrt{0.360}} = \frac{1}{0.600} = 1.667.$$

The ship will shrink by this amount, according to a stationary observer, but only in the direction of motion (along its length). Its length will appear to be

$$L = \frac{L_p}{\gamma} = \frac{260 \text{ m}}{1.667} = 156 \text{ m}.$$

However, its diameter will remain unchanged at 21.0 m.

b) **The spaceship sends a signal to Earth every one-sixth of a cycle around its orbit at a frequency of 225 MHz. What is the frequency of the received signal coming from points *A*, *B*, *C*, and *D*?**

The frequency of the detected signal is given by

$$f = \frac{f_0}{\gamma(1-v\cos\theta/c)} = \frac{225 \text{ MHz}}{1.667(1-0.8\cos\theta)} = \frac{135 \text{ MHz}}{1-0.8\cos\theta}$$

We now just plug in the four angles, 0°, 60°, 120°, and 180° to give

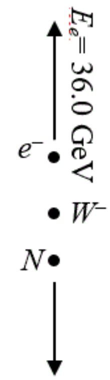
$$f_A = \frac{135 \text{ MHz}}{1-0.800\cos(0°)} = \frac{135 \text{ MHz}}{1-0.800} = 675 \text{ MHz},$$

$$f_B = \frac{135 \text{ MHz}}{1-0.800\cos(60°)} = \frac{135 \text{ MHz}}{1-0.400} = 225 \text{ MHz},$$

$$f_C = \frac{135 \text{ MHz}}{1-0.800\cos(120°)} = \frac{135 \text{ MHz}}{1+0.400} = 96.4 \text{ MHz},$$

$$f_D = \frac{135 \text{ MHz}}{1-0.800\cos(180°)} = \frac{135 \text{ MHz}}{1+0.800} = 75.0 \text{ MHz}.$$

15. A W^- particle at rest (m_W = 80.4 GeV/c^2) decays to a nearly massless electron moving up with energy 36.0 GeV ($m_e \approx 0$), and a massive neutrino N moving down.



a) Find the energy (in GeV) and momentum (in GeV/c) of the N.

The W is at rest, so its momentum is zero and its energy is

$$E_W = m_W c^2 = 80.4 \text{ GeV}$$

The electron is approximately massless, so we can use the simplified formula $E = c\left|\vec{p}\right|$ to find its momentum, $\left|\vec{p}\right| = E/c = 36.0 \text{ GeV}/c^2$. To be on the safe side, we will recall that momentum is a vector, and since it's moving in the y-direction, we write its momentum as

$$\vec{p}_e = 36.0\hat{\mathbf{j}} \text{ GeV}/c.$$

By conservation of momentum and energy, the energy and momentum of the W must match the electron and the N, so we have

$$E_N = E_W - E_e = (80.4 \text{ GeV}) - (36.0 \text{ GeV}) = 44.4 \text{ Gev},$$

$$\vec{p}_N = \vec{p}_W - \vec{p}_e = 0 - \left(36.0\hat{\mathbf{j}} \text{ GeV}/c\right) = -36.0\hat{\mathbf{j}} \text{ GeV}/c.$$

b) Find the mass (in GeV/c^2) and speed (as a fraction of c) of the N.

The mass can be found most easily from

$$\left(m_N c^2\right)^2 = E_N^2 - \left(\vec{p}_N c\right)^2 = (44.4 \text{ GeV})^2 - \left(-36.0\hat{\mathbf{j}} \text{ GeV}\right)^2 = 675.4 \text{ GeV}^2,$$

$$m_N c^2 = 26.0 \text{ GeV}.$$

The velocity is given by

$$\frac{\vec{u}}{c} = \frac{\vec{p}c}{E} = \frac{-36.0\hat{\mathbf{j}} \text{ GeV}}{44.4 \text{ GeV}} = -0.811\hat{\mathbf{j}}.$$

So the mass is 26.0 GeV/c^2 and the speed is $0.811c$.

16. A nanobot is launched from rest to explore the stars! It has a mass of $m = 1.62\times10^{-9}$ kg, and is intended to travel to the stars at a speed of $v = 2.10\times10^{8}$ m/s. It will be accelerated electromagnetically with a force F = 13.75 N.

a) What are the initial and final momentum of the nanobot? What are the initial and final energy of the nanobot?

The nanobot is initially at rest, so its momentum is zero. Its final momentum is given by

$$p_f = \gamma_f m u_f = \frac{\left(1.62\times10^{-9}\text{ kg}\right)\left(2.10\times10^{8}\text{ m/s}\right)}{\sqrt{1-\left(\dfrac{2.10\times10^{8}\text{ m/s}}{2.998\times10^{8}\text{ m/s}}\right)^2}} = \frac{0.3402\text{ kg}\cdot\text{m/s}}{\sqrt{1-0.7005^2}} = 0.477\,\text{kg}\cdot\text{m/s}.$$

The change in momentum is, of course, the same as the final momentum.

The nanobot starts at rest, so its initial energy is $E_i = mc^2$, and its final energy is $E_f = \gamma mc^2$, so we have

$$E_i = mc^2 = \left(1.62\times10^{-9}\text{ kg}\right)\left(2.998\times10^{8}\text{ m/s}\right)^2 = 1.456\times10^{8}\text{ J},$$

$$E_f = \gamma_f mc^2 = \frac{\left(1.62\times10^{-9}\text{ kg}\right)\left(2.998\times10^{8}\text{ m/s}\right)^2}{\sqrt{1-\left(\dfrac{2.10\times10^{8}\text{ m/s}}{2.998\times10^{8}\text{ m/s}}\right)^2}} = \frac{1.456\times10^{8}\text{ J}}{\sqrt{1-0.7005^2}} = 2.040\times10^{8}\text{ J}.$$

b) How long will it take to accelerate the nanobot to this speed?

We use the formula $F = \dfrac{\Delta p}{\Delta t}$ and solve for the time:

$$\Delta t = \frac{\Delta p}{F} = \frac{0.477\text{ kg}\cdot\text{m/s}}{13.75\text{ kg}\cdot\text{m/s}^2} = 0.0347\text{ s}.$$

We were not asked for the distance, but it can be found from $\Delta E = W = Fd$, or solving for d:

$$d = \frac{\Delta E}{F} = \frac{\left(2.040\times10^{8}\text{ J}\right)-\left(1.456\times10^{8}\text{J}\right)}{13.75\text{ N}} = 4.25\times10^{6}\text{ m} = 4250\text{ km}.$$