Solutions to Test 1
September 15, 2021

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered.

Possibly useful formulas:

$$x' = \gamma(x - vt)$$
$$t' = \gamma(t - vx/c^2)$$
$$y' = y, \quad z' = z$$

$$E' = \gamma(E - vp_x)$$
$$p'_x = \gamma(p_x - vE/c^2)$$
$$p'_y = p_y, \quad p'_z = p_z$$

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2}$$
$$u'_y = \frac{u_y}{\gamma(1 - vu_x/c^2)}$$
$$u'_z = \frac{u_z}{\gamma(1 - vu_x/c^2)}$$

$$f = \frac{f_0}{\gamma(1 - v\cos\theta/c)}$$

$$\vec{F} = q\vec{\mathrm{E}} + q\vec{u} \times \vec{\mathrm{B}}$$
$$p = qRB$$

$$\frac{\vec{u}}{c} = \frac{\vec{p}c}{E}$$

$$e = 1.602 \times 10^{-19} \text{ C}$$
$$\text{eV} = 1.602 \times 10^{-19} \text{ J}$$
$$\text{y} = 3.156 \times 10^{7} \text{ s}$$
$$\text{ly} = 9.461 \times 10^{15} \text{ m}$$

Part I: Multiple Choice [20 points]
For each question, choose the best answer (2 points each)

1. If two meter sticks are moving past each other in the direction of length, which one will be shorter?
   A) The one that is actually moving is shorter
   B) The one that is actually stationary is shorter
   C) Each will claim that the other meter stick is shorter
   D) Each will claim that the other meter stick is longer
   E) Since both are meter sticks, they will be the same

2. If one person travels at high velocity to a distant star and returns, and another stays on Earth the whole time, when they are reunited which one will apparently be older?
   A) They will be the same age
   B) The one who travels is older
   C) The one on Earth is older
   D) It depends on whether the person who traveled went in a straight line or not
   E) I have no idea; please mark this one wrong

3. At some point you received a letter indicating you were admitted to Wake Forest, and now you are here. How does the receipt of that letter compare to here and now?
   A) It was in the absolute past
   B) It was on the past light cone
   C) It is in the past according to us, but another observer might say it was in the future
   D) It was in the absolute future
   E) Insufficient information

4. If a particle had a negative mass squared (a tachyon), what property would it also have?
   A) It would always have negative energy
   B) It would always travel faster than light
   C) It would have zero size
   D) It would exist only at one moment in time
   E) It would have an imaginary velocity

5. A mass spectrometer has causes particles to move in a circle. How is this managed?
   A) Constant magnetic fields
   B) Constant electric fields
   C) Oscillating magnetic fields
   D) Oscillating electric fields
   E) Circular wave guides

6. In the limit that the velocity of a massive object approaches the speed of light, the energy will approach _______ and the momentum will approach _______ .
   A) mc^2, mc B) $\frac{1}{2}mc^2, mc$ C) mc^2, ∞ D) ∞, mc **E)** ∞, ∞

7. Which type(s) of energy, when added to an object, will increase its mass?
   A) Nuclear energy (only)
   B) Chemical energy (only)
   C) Heat energy (only)
   D) All of the above
   E) None of the above

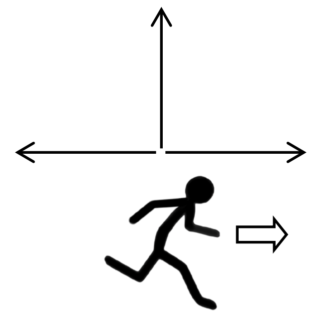
8. Since space-time is four dimensional, momentum should have a fourth component. What is the fourth component of momentum?
   A) Mass B) Time C) Distance D) Velocity **E) Energy**

9. There were four transformations that we described as "good" in class: space translation, time translation, rotation, and Lorentz boost. What distinguishes these transformations from "bad" ones like the Galilean boost or the rescaling transformation?
   A) They were experimentally observed to be correct before Einstein
   B) They preserve the four-dimensional distance formula
   C) Objects moving at constant velocity will still be at constant velocity
   D) They lead to the correct formulas for momentum in relativity
   E) They take into account that moving objects have more energy

10. Which of the following formulas is still valid in special relativity, without modification?
   A) $E_{kin} = \frac{1}{2}mv^2$ B) $F = ma$ C) $p = mv$ **D)** $W = Fd$ E) $F = \frac{d}{dt}(mv)$

Part II: Short answer [20 points]

Choose **two** of the following questions and give a short answer (1-3 sentences) (10 points each).



11. Three light beams are moving to the right, up, and to the left, each with velocity *c*. According to an observer moving to the right at 0.5*c*, which, if any, of the three light beams will appear to move faster than *c*, and which slower than *c*?

The speed of light in vacuum is given by *c*, according to all observers. The fact that the observer is moving to the right does not change this for any of the three light beams. The one pointed up will now point backwards at an angle, but its speed is still *c*.

12. In a variation of the barn-pole paradox, a runner runs completely through the barn (in the front, out the back). The runner claims both ends of the pole simultaneously sticks out of the barn, while the farmer claims both ends of the pole are simultaneously inside the barn. Explain how special relativity helps resolve this apparent paradox.

The key to understanding this puzzle is simultaneity. To determine if the pole is inside the barn, you need to measure everything's position simultaneously, but different observers will disagree on what is simultaneous.

13. Neutral hydrogen atoms are detected by their radiation from the hyperfine splitting with a natural frequency of $f = 1420.4$ MHz; however, the frequency detected is often lower or higher than this frequency. Explain how the frequency shifts if the hydrogen cloud is moving (i) towards us, (ii) away from us, or (iii) perpendicular to our line of sight. If two of them are affected the same way, make clear which of them is the larger effect (for fixed speed).

If the cloud is moving towards us, it will be blue-shifted to higher frequency, and if it is moving away from us it will be red-shifted to lower frequency. If it is moving perpendicular to our line of sight, it will also be red-shifted, but not as much as if it were moving away from us.

Part III: Calculation: [60 points]
Choose **three** of the following four questions and perform the indicated calculations (20 points each)

14. Consider three events in space-time with coordinates (x, y, z, ct) given by

$$A = (0,0,0,0)\,\mathrm{m}\ ; \quad B = (2,2,0,3)\,\mathrm{m}\ ; \quad C = (3,4,0,5)\,\mathrm{m}.$$

For each of the pairs listed below, determine whether the separation is timelike, spacelike, or lightlike, and when appropriate, determine which is in the past/future of the other (absolute past/future, or past/future light cone). If the separation is not lightlike, also find either the proper distance *s* or the proper time $c\tau$ in meters.
(a) *A* with *B* **(b) *A* with *C*** **(c) *B* with *C***

We will repeatedly use the proper distance/proper time formula:

$$s^2 = -c^2\tau^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2$$

Applying it in each case, we have

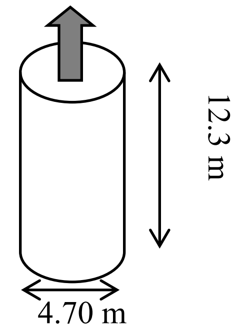
$$s_{AB}^2 = \left[(2-0)^2 + (2-0)^2 + (0-0)^2 - (3-0)^2\right]\mathrm{m}^2 = (4+4-9)\,\mathrm{m}^2 = -1\ \mathrm{m}^2\ ,$$
$$s_{AC}^2 = \left[(3-0)^2 + (4-0)^2 + (0-0)^2 - (5-0)^2\right]\mathrm{m}^2 = (9+16-25)\,\mathrm{m}^2 = 0\ \mathrm{m}^2\ ,$$
$$s_{BC}^2 = \left[(3-2)^2 + (4-2)^2 + (0-0)^2 - (5-3)^2\right]\mathrm{m}^2 = (1+4-4)\,\mathrm{m}^2 = 1\ \mathrm{m}^2\ .$$

The first one came out negative, so the separation is timelike. Since *B* has a larger time coordinate than *A*, *B* is in the absolute future of *B*. Since $-c^2\tau^2 = -1\ \mathrm{m}^2$, $c\tau = 1\ \mathrm{m}$ is the proper time.

The second one came out zero, so the separation is lightlike. Since *C* has a larger time coordinate than *A*, *C* is in the absolute future of *A*. Since the result is zero, we don't need to take the square root

The third one came out positive, so the separation is spacelike. Though *C* has the larger time coordinate, other observers will disagree, so neither is unambiguously in the future or past of the other. Since $s^2 = 1\ \mathrm{m}^2$, $s = 1\ \mathrm{m}$ is the proper distance.

15. Travelers board a cylindrical spacecraft which is 12.3 m long in the direction it is moving and 4.70 m in diameter in the perpendicular direction. The spacecraft is traveling to the star Sirius which is 8.61 ly (= 8.61 *c·y*) away. They will travel at *v* = 2.18×10^8 m/s



(a) What are the dimensions of the spacecraft as viewed by us?

The spacecraft is travelling with a Lorentz factor given by

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\left(\frac{2.18\times10^8\text{ m/s}}{2.998\times10^8\text{ m/s}}\right)^2}} = \frac{1}{\sqrt{1-0.727^2}} = 1.456.$$

Because spacecraft is traveling, its length will be Lorentz contracted according to the formula

$$L = \frac{L_p}{\gamma} = \frac{12.3\text{ m}}{1.457} = 8.44\,\text{m}.$$

(b) How long in years will it take to get there according to us?

This is straightforward and does not involve relativity, $d = v\Delta t$, so

$$\Delta t = \frac{d}{v} = \frac{8.61c\cdot\text{y}}{2.18\times10^8\text{ m/s}} = (8.61\text{ y})\frac{2.998\times10^8\text{ m/s}}{2.18\times10^8\text{ m/s}} = 11.84\text{ y}.$$

(c) How long in years will it take to get there according to the occupants?

Due to the high speed, the proper time τ will be related to the time Δt by $\Delta t = \gamma\tau$, so

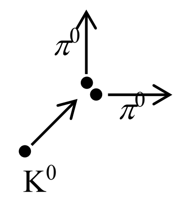
$$\tau = \frac{\Delta t}{\gamma} = \frac{11.84\text{ y}}{1.456} = 8.13\text{ y}.$$

(d) If the total spacecraft has a mass of 5430 kg, what is the kinetic energy of the spacecraft in J?

The kinetic energy is the total energy minus the rest energy, so

$$E = \gamma mc^2 - mc^2 = (\gamma-1)mc^2 = 0.456(5430\text{ kg})(2.998\times10^8\text{ m/s})^2 = 2.226\times10^{20}\text{ J}.$$

16. A K^0 meson decays to two pions with mass $m_\pi = 135.0$ MeV/c^2, one of which goes to the right and one of which goes straight up, both at a speed of $u = 0.9078c$.



(a) What is the energy of each of the pions (in MeV)?

The energy is related to the mass and speed by $E = \gamma mc^2$, so we have

$$E_1 = E_2 = \gamma m_\pi c^2 = \frac{m_\pi c^2}{\sqrt{1-(u/c)^2}} = \frac{\left(135.0 \text{ MeV}/c^2\right)c^2}{\sqrt{1-0.9078^2}} = 321.9 \text{ MeV}.$$

(b) What is the momentum of each of the pions (in MeV/c)?

The momentum is given by $p = \gamma mu$, so

$$p = \gamma mu = \frac{\left(135.0 \text{ MeV}/c^2\right)(0.9078c)}{\sqrt{1-0.9078^2}} = 292.3 \text{ MeV}/c.$$

However, momentum is a vector, so for the one going to the right we have $\vec{p}_1 = 292.3\hat{\mathbf{i}} \text{ MeV}/c$, and for the other $\vec{p}_2 = 292.3\hat{\mathbf{j}} \text{ MeV}/c$.

(c) What is the energy and momentum of the K^0 before it decays?

The energy and momentum of the K^0 is just the sum of the energy and momenta of the pions, so

$$E_K = E_1 + E_2 = 2(321.9 \text{ MeV}) = 643.8 \text{ MeV},$$
$$\vec{p}_K = \vec{p}_1 + \vec{p}_2 = \left(292.3\hat{\mathbf{i}} + 292.3\hat{\mathbf{j}}\right) \text{ MeV}/c.$$

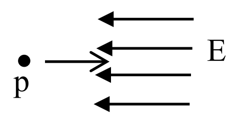
(d) What is the mass of the K^0 (in MeV/c^2)?

We use the formula

$$\left(mc^2\right)^2 = E^2 - c^2\vec{p}^2 = E^2 - c^2p_x^2 - c^2p_y^2 - c^2p_z^2$$
$$= (643.8 \text{ MeV})^2 - (292.3 \text{ MeV})^2 - (292.3 \text{ MeV})^2 - 0 = 243600 \text{ MeV}^2,$$
$$mc^2 = \sqrt{243600 \text{ MeV}^2} = 493.6 \text{ MeV}.$$

So the mass is 493.6 MeV/c^2.

17. A proton of mass $m = 938\ \text{MeV}/c^2$ and charge $+e = 1.602\times10^{-19}$ C is moving to the right at a speed of $u = 0.468c$. It is being pushed to the left by a constant electric field of $\text{E} = 1.602\times10^{6}$ V/m,



(a) What is its initial momentum, in MeV/*c*?

The momentum is given by

$$p = \gamma m v = \frac{\left(938\ \text{MeV}/c^2\right)\left(0.468c\right)}{\sqrt{1-0.468^2}} = 496.7\ \text{MeV}/c.$$

(b) How long will it take to come to a stop, in s?

We first need the force, which is

$$F = q\text{E} = e\text{E} = 1.602\times10^{6}\ \text{eV/m} = 1.602\ \text{MeV/m}.$$

We can then get the time using the formula $\Delta p = F\Delta t$, which we solve for the time

$$\Delta t = \frac{\Delta p}{F} = \frac{496.7\ \text{MeV}/c}{1.602\ \text{MeV/m}} = \frac{310\ \text{m}}{c} = \frac{310\ \text{m}}{2.998\times10^{8}\ \text{m/s}} = 1.034\times10^{-6}\ \text{s} = 1.034\ \mu\text{s}.$$

(c) After the proton stops it will turn around and go the opposite direction. When it has returned to its starting point, what will be its velocity (direction and sign) be? (Hint: use the work formula). How long will it take to return to this point?

The total displacement is zero, since it is back at the original point, so by the formula $W = Fd$, there is no work on the particle and hence the final energy must match the original. Hence the final *speed* must be the same, but it is moving to the left now, so $u = -0.468c$.

We can then repeat the work of part (b). Since the velocity is the negative of the starting momentum, the change in the momentum will be twice what it was before, and hence the time will be doubled, so in this case

$$\Delta t = \frac{\Delta p}{F} = \frac{2\left(496.7\ \text{MeV}/c\right)}{1.602\ \text{MeV/m}} = \frac{2\cdot310\ \text{m}}{c} = \frac{2\cdot310\ \text{m}}{2.998\times10^{8}\ \text{m/s}} = 2.069\times10^{-6}\ \text{s} = 2.069\ \mu\text{s}.$$