Name ___________________

Solutions to Test 2
October 19, 2018

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered. The equations below may be helpful with some problems.

Constants

$h = 6.626\times10^{-34}\ \text{J}\cdot\text{s}$

$h = 4.136\times10^{-15}\ \text{eV}\cdot\text{s}$

$\hbar = 1.055\times10^{-34}\ \text{J}\cdot\text{s}$

$\hbar = 6.582\times10^{-16}\ \text{eV}\cdot\text{s}$

$k_B = 1.3807\times10^{-23}\ \text{J/K}$

$k_B = 8.6173\times10^{-5}\ \text{eV/K}$

$k = 8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2$

$e = 1.602\times10^{-19}\ \text{C}$

$1\ \text{eV} = 1.602\times10^{-19}\ \text{J}$

$\alpha = \dfrac{ke^2}{\hbar c} \simeq \dfrac{1}{137}$

Hydrogen-Like Atoms

$$E = -\frac{k^2e^4\mu Z^2}{2\hbar^2n^2} = -\frac{(\mu c^2)\alpha^2Z^2}{2n^2}$$

$$E = \frac{-(13.60\ \text{eV})Z^2}{n^2}$$

Hydrogen Spectrum

$$\lambda = (91.17\ \text{nm})\left(\frac{1}{n^2}-\frac{1}{m^2}\right)^{-1}$$

Black Bodies

$$U = \frac{\pi^2(k_BT)^4}{15(\hbar c)^3}$$

$$\lambda_{\max}T = .002898\ \text{m}\cdot\text{K}$$

Wave Relationships

$$\lambda = \frac{2\pi}{k}$$

$$\frac{\omega}{2\pi} = f = \frac{1}{T}$$

Reduced Mass

$$\mu = \frac{mM}{m+M}$$

Compton Effect

$$\lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$$

$$\frac{h}{mc} = 2.426\times10^{-12}\ \text{m}$$

Rutherford Scattering

$$b = \frac{kqQ}{m_\alpha v^2}\cot\left(\frac{\theta}{2}\right)$$

$$R = \frac{2Ze^2k}{E}$$

Part I: Multiple Choice [20 points]

For each question, choose the best answer (2 points each)

1. The ratio of the charge to mass of an electron was measured by J. J. Thomson by measuring
   A) How the electron curved in the presence of an electric field
   B) How the electorn curved in the presence of a magnetic field
   C) The scattering of alpha-particles off of nuclei
   D) The change in mass of an oil droplet as it picked up or lost an electron
   E) The quantity of charge required to dissociate a mole of various compounds in solution

2. In the photoelectric effect, which determines how much energy an electron ejected from the metal has?
   A) The polarization of the light
   B) The intensity (brightness) of the light
   C) The direction the light impinges the metal at
   D) The frequency of the light
   E) The size of the electric field in the light

3. The speed at which a wave packet as a whole moves is called the _________ velocity, while the speed at which the peak of an individual wave moves is called the __________ velocity
   A) Group, phase B) Phase, group C) Packet, peak D) Lump, wave E) Bunch, wave

4. Which of the following equals $\cos\theta - i\sin\theta$?
   A) $e^{i\theta}$ B) $-e^{i\theta}$ **C)** $e^{-i\theta}$ D) $-e^{-i\theta}$ E) None of these

5. Wave properties of objects passing through an aperture (like a slit) are most likely to be noticeable when
   A) The wavelength is large compared to the aperture
   B) The wavelength is small compared to the aperture
   C) The frequency is much lower than the speed of the particles
   D) The frequency is much higher than the speed of the particles
   E) The waves are generated by a wave generator, rather than a particle generator.

6. The de Broglie relationship between wavelength and momentum is
   A) $p = h\lambda$ B) $\lambda = hp$ C) $\lambda hp = 1$ **D)** $\lambda p = h$ E) None of these

7. Find the partial derivative $\frac{\partial}{\partial x}\left(x^2 - y^2\right)$
   A) $2x - 2y$ B) $2x - y^2$ C) $-2y$ D) x^2 **E)** $2x$

8. In which situation is the Bohr model not successful in describing the energy levels?
   A) An hydrogen atom
   B) A non-hydrogen atom with one electron
   C) A many-electron atom describing the innermost electrons
   D) A many-electron atom describing the outermost electrons
   E) Actually, it applies to all of these cases

9. Which of the following are believed to not have quantum mechanical properties, that is, they do not have wave-like properties
   A) Electrons
   B) Protons
   C) Atoms
   D) Molecules
   E) They actually all have wave properties

10. How did the scattering measurements of Rutherford of alpha-particle off of atoms change our picture of the atom?
   A) We realized that alpha-particles were in fact, very large, rather than very small
   B) We learned that electrons are not present in the nucleus of the atom
   C) We learned that the positive charge is concentrated in a small regoin called the nucleus
   D) We learned that the electrons are attracted to the nucleus
   E) We learned the approximate size of atoms

Part II: Short answer [20 points]

Choose **two** of the following questions and give a short answer (2-3 sentences) (10 points each).

11. In the photoelectric effect, explain why it is that a minimum frequency/maximum wavelength is required to free electrons. At least one formula would be helpful.

To get the electron free, the energy of the incoming photon must exceed the work function ϕ required to get it free. Since the energy of a photon is $E = hf$, we therefore need $hf > \phi$. Since frequency and wavelength are inversely proportional, from $f\lambda = c$, a minimum frequency also corresponds to a maximum wavelength.

12. The harmonic oscillator has potential energy function $V(x) = \frac{1}{2}m\omega^2 x^2$, which obviously has a minimum value of $V(x=0) = 0$. Nonetheless, the actual minimum energy for the harmonic oscillator turns out to be positive, $E_{\min} > 0$. Explain qualitatively why achieving zero energy is in fact impossible for the harmonic oscillator.

In order to be at the minimum energy, we would have to set both the position equal to zero and the momentum equal to zero. But for a quantum system, it is impossible to control both of these at the same time, as they are governed by the uncertainty principle, $(\Delta x)(\Delta p) \geq \frac{1}{2}\hbar$. This causes there to be a minimum energy, or zero-point energy.

13. Atomic nuclei are not measured by placing tiny calipers near them; instead, they are measured by measuring the cross-section for scattering. Explain qualitatively what cross-section is. If you were firing a shotgun at a sphere of radius 1.00 cm, what would be the cross-section for hitting the sphere?

The cross-section represents how large the effective surface is from which the particles can scatter. In particular, if you are trying to hit a sphere of radius 1.00 cm, from one side it looks like a circle with this radius, so the cross-section is $\pi r^2 = \pi \text{ cm}^2$.

Part III: Calculation: [60 points]

Choose **three** of the following four questions and perform the indicated calculations (20 points each).

14. In the early universe, there was a time when protons and electrons first came together to form hydrogen atoms. This era is called *recombination*, and it occurred at a temperature of approximately $T = 2976$ K. It is believed that the radiation at this time was almost a perfect black body radiation

(a) What wavelength of light λ_{max} would have been the peak of the spectrum at this time?

We use Wien's Law, $\lambda_{\text{max}} T = .002898\ \text{m}\cdot\text{K}$, and solve for λ_{max} to give

$$\lambda_{\text{max}} = \frac{.002898\ \text{m}\cdot\text{K}}{T} = \frac{.002898\ \text{m}\cdot\text{K}}{2976\ \text{K}} = 9.73\times10^{-7}\ \text{m} = 973\ \text{nm}.$$

(b) What is the corresponding frequency (in Hz) and energy (in eV) for a photon with the wavelength you found in (a)?

We find the frequency using $c = f\lambda$, which we solve for *f* to yield

$$f = \frac{c}{\lambda} = \frac{2.998\times10^{8}\ \text{m/s}}{9.73\times10^{-7}} = 3.08\times10^{14}\ \text{Hz}.$$

We can then get the energy from $E = hf$, so

$$E = hf = \left(4.136\times10^{-15}\ \text{eV}\cdot\text{s}\right)\left(3.08\times10^{14}\ \text{s}^{-1}\right) = 1.274\ \text{eV}.$$

(c) What was the energy density (in J/m³) of the black-body radiation at that time?

For this we use the more complicated formula for the energy density, namely

$$u = \frac{\pi^2 (k_B T)^4}{15(\hbar c)^3} = \frac{\pi^2\left[\left(1.3807\times10^{-23}\ \text{J/K}\right)\left(2976\ \text{K}\right)\right]^4}{15\left[\left(1.055\times10^{-34}\ \text{J}\cdot\text{s}\right)\left(2.998\times10^{8}\ \text{m/s}\right)\right]^3} = 0.0593\ \text{J/m}^3.$$

15. A tau-minus lepton (τ^-) is much like an electron, but much heavier. It has a mass of $m_\tau = 1776.8\text{ MeV}/c^2$, and can bind to a proton with a mass of $m_p = 938.3\text{ MeV}/c^2$.

(a) What is the reduced mass for this system?

The reduced mass is given by the formula

$$\mu = \frac{m_p m_\tau}{m_p + m_\tau} = \frac{\left(938.3\text{ MeV}/c^2\right)\left(1776.8\text{ MeV}/c^2\right)}{\left(938.3\text{ MeV}/c^2\right) + \left(1776.8\text{ MeV}/c^2\right)} = 614.04\text{ MeV}/c^2 .$$

(b) Find a formula for the energy of the *n*'th state for this system, in keV.

The proton and tau both have charge one, so we have $Z = 1$. We now use the general formula, namely

$$E_n = -\frac{\left(\mu c^2\right)\alpha^2 Z^2}{2n^2} = -\frac{614.04\text{ MeV}}{2\left(137\right)^2 n^2} = -\frac{0.01636\text{ MeV}}{n^2} = -\frac{16.36\text{ keV}}{n^2} .$$

(c) An atom in this system goes from the *n* = 25 state to the *n* = 24 state. What is the energy in eV of the photon released in the process?

The energy is simply the initial energy minus the final energy, so

$$\Delta E = E_{25} - E_{24} = -\frac{16.36\text{ keV}}{25^2} + \frac{16.36\text{ keV}}{24^2} = 0.00223\text{ keV} = 2.23\text{ eV} .$$

16. An X-ray photon with unknown wavelength scatters from a stationary target. It is found that for those X-rays that are scattered at an angle of 60°, the scattered X-rays have a wavelength of $14.74\text{ pm} = 14.74\times10^{-12}\text{ m}$

(a) What was the wavelength of the photons before they were scattered?

We use the Compton effect formula, $\lambda' - \lambda = (1-\cos\theta)(h/mc)$, where $h/mc = 2.426\text{ pm}$. Solving for λ, we have

$$\lambda = \lambda' - (1-\cos\theta)(h/mc) = (14.74\text{ pm}) - (1-\cos 60°)(2.426\text{ pm}) = 13.53\text{ pm}.$$

(b) What is the wavelength of the photons that are scattered by 180°?

We just use the same formula in its most straightforward manner, using the wavelength computed in part (a):

$$\lambda' = \lambda + (1-\cos\theta)(h/mc) = (13.53\text{ pm}) + (1-\cos 180°)(2.426\text{ pm}) = 18.38\text{ pm}.$$

(c) At what angle θ will the scattered wavelength be $17.10\text{ pm} = 17.10\times10^{-12}\text{ m}$**?**

This just requires a bit more cleverness, since we have to solve for the angle θ:

$$(1-\cos\theta)(h/mc) = \lambda' - \lambda = (17.10\text{ pm}) - (13.53\text{ pm}) = 3.57\text{ pm},$$

$$1-\cos\theta = \frac{3.57\text{ pm}}{(h/mc)} = \frac{3.57\text{ pm}}{2.426\text{ pm}} = 1.472,$$

$$\cos\theta = 1 - 1.472 = -0.472,$$

$$\theta = \cos^{-1}(-0.472) = 118°.$$

17. The wave function for a particle is given by $\psi(x)=\begin{cases} N\left(e^{-\lambda x}-e^{-2\lambda x}\right) & x\ge 0\,, \\ 0 & x<0\,, \end{cases}$ **where** λ **is a positive constant. A possibly helpful integral is given below.**

(a) Where in the allowed region $x \ge 0$ **is the particle most likely to be?**

The place where the particle is most likely to be is when the wave function is most positive, or most negative, which must be at a maximum or minimum. Hence we simply set the derivative equal to zero, which gives us

$$0=\frac{d}{dx}\psi(x)=N\frac{d}{dx}\left(e^{-\lambda x}-e^{-2\lambda x}\right)=N\left(-\lambda e^{-\lambda x}+2\lambda e^{-2\lambda x}\right),$$

$$e^{-\lambda x}=2e^{-2\lambda x}\,,$$

$$e^{\lambda x}=2,$$

$$\lambda x=\ln(2),$$

$$x=\lambda^{-1}\ln(2).$$

(b) What is the normalization constant *N*?

We demand that the integral of $|\psi|^2$ be one, so we have

$$1=\int_{-\infty}^{\infty}|\psi|^2\,dx=N^2\int_0^{\infty}\left(e^{-\lambda x}-e^{-2\lambda x}\right)^2dx=N^2\int_0^{\infty}\left(e^{-2\lambda x}-2e^{-3\lambda x}+e^{-4\lambda x}\right)dx=N^2\left(\frac{1}{2\lambda}-\frac{2}{3\lambda}+\frac{1}{4\lambda}\right)$$

$$=\frac{N^2}{12\lambda},$$

$$N=\sqrt{12\lambda}\,.$$

(c) If the particle's position is measured, what is the probability that it will have $x>\lambda^{-1}$**?**

We simply repeat the same computation, but this time change the lower limit to λ^{-1}.

$$P\left(x>\lambda^{-1}\right)=N^2\int_a^{\infty}\left(e^{-2\lambda x}-2e^{-3\lambda x}+e^{-4\lambda x}\right)dx=12\lambda\left(\frac{e^{-2\lambda/\lambda}}{2\lambda}-\frac{2e^{-3\lambda/\lambda}}{3\lambda}+\frac{e^{-4\lambda/\lambda}}{4\lambda}\right)$$

$$=6e^{-2}-8e^{-3}+3e^{-4}\approx 0.469=46.9\%\,.$$

Possibly helfpul integral: $\int_\alpha^\infty e^{-Ax}dx=e^{-A\alpha}/A$