Solutions to Test 2
October 16, 2019

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered. The equations below may be helpful with some problems.

Constants	Hydrogen-Like Atoms	Hydrogen Spectrum
$h = 6.626\times10^{-34}\ \text{J}\cdot\text{s}$ $h = 4.136\times10^{-15}\ \text{eV}\cdot\text{s}$ $\hbar = 1.055\times10^{-34}\ \text{J}\cdot\text{s}$ $\hbar = 6.582\times10^{-16}\ \text{eV}\cdot\text{s}$ $k_B = 1.3807\times10^{-23}\ \text{J/K}$ $k_B = 8.6173\times10^{-5}\ \text{eV/K}$ $k = 8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2$ $e = 1.602\times10^{-19}\ \text{C}$ $1\ \text{eV} = 1.602\times10^{-19}\ \text{J}$ $\alpha = \dfrac{ke^2}{\hbar c} \simeq \dfrac{1}{137}$	$E = -\dfrac{k^2e^4\mu Z^2}{2\hbar^2 n^2} = -\dfrac{(\mu c^2)\alpha^2 Z^2}{2n^2}$ $E = \dfrac{-(13.60\ \text{eV})Z^2}{n^2}$	$\lambda = (91.17\ \text{nm})\left(\dfrac{1}{n^2}-\dfrac{1}{m^2}\right)^{-1}$

Wave Relationships	Reduced Mass	Black Bodies
$\lambda = \dfrac{2\pi}{k}$ $\dfrac{\omega}{2\pi} = f = \dfrac{1}{T}$	$\mu = \dfrac{mM}{m+M}$	$U = \dfrac{\pi^2 (k_B T)^4}{15(\hbar c)^3}$ $\lambda_{\max} T = .002898\ \text{m}\cdot\text{K}$

Compton Effect	Rutherford Scattering
$\lambda' - \lambda = \dfrac{h}{mc}(1-\cos\theta)$ $\dfrac{h}{mc} = 2.426\times10^{-12}\ \text{m}$	$b = \dfrac{kqQ}{m_\alpha v^2}\cot\left(\dfrac{\theta}{2}\right)$ $R = \dfrac{2Ze^2k}{E}$

Part I: Multiple Choice [20 points]
For each question, choose the best answer (2 points each)

1. Which of the following is not a basic wave function?
   A) $\sin(kx-\omega t)$ B) $\cos(kx-\omega t)$ C) $e^{i(kx-\omega t)}$ **D) $e^{kx^2-i\omega t}$** E) Actually, they all are fine

2. How does the peak wavelength of light from a blackbody change as the temperature changes
   A) It is always concentrated at the longest possible wavelengths
   B) It is always concentrated at the shortest possible wavelengths
   C) The peak wavelength is independent of temperature
   D) The peak wavelength is directly proportional to the temperature
   E) The peak wavelength is inversely proportional to the temperature

3. Why is it that we don't normally notice wave properties of objects like cars and people?
   A) The wavelength is so short that effectively you can't see the effects
   B) The wavelength is so large that the oscillations caused by waves are invisible
   C) Composite objects like cars and people are believed to not have wave properties
   D) Because of the uncertainty principle, we can't be certain if they have wave properties or not
   E) They can only be detected by interacting them with other waves, not with objects

4. Simplify $e^{-i\pi/2}$
   A) 1 B) -1 C) i **D) $-i$** E) None of these

5. Quantum mechanically, we had two uncertainty relationships, one about position and one about time. What does the one about time imply?
   A) Measuring the momentum of a particle very precisely makes it very uncertain at what time the measurement was done
   B) The velocity of a particle is generally uncertain, since there is uncertainty in the time it took to go any given distance
   C) A particle or other quantum state with a finite lifetime will have an uncertainty in its energy
   D) The time it takes for something to occur in quantum mechanics is directly proportional to how much energy is required
   E) I can't be certain both that I have the energy and the time to finish this exam

6. Electrons scatter off of crystals especially strongly when they match the Bragg condition, so that electrons scattering off of one layer interfere constructively with other layers. This experiment demonstrates that electrons have
   A) Wave properties B) Particle properties C) Negative charge D) Momentum E) Energy

7. If we knew the relationship between the angular frequency ω and the wave number k, which of the following formulas would give the group velocity, the speed at which the wave packets travel?
   A) $\frac{\omega}{k}$ B) $\frac{k}{\omega}$ **C)** $\frac{d\omega}{dk}$ D) $\frac{dk}{d\omega}$ E) None of these

8. Suppose the uncertain in the position of a particle is $\Delta x = a$. What can we say about the uncertainty in the momentum Δp?
   A) $\Delta p \le \frac{\hbar}{2a}$ **B)** $\Delta p \ge \frac{\hbar}{2a}$ C) $\Delta p \le \frac{h}{2a}$ D) $\Delta p \ge \frac{h}{2a}$ E) None of these

9. Which of the following might be the ratio of the size of an atom divided by the size of the nucleus?
   A) 300 **B) 3×10^4** C) 3×10^6 D) 3×10^8 E) 3×10^{10}

10. To which particles does the de Broglie relationship $\lambda p = h$ apply?
    A) Photons (only)
    B) Electrons (only)
    C) Photons and electrons, but not protons
    D) Photons, electrons, and protons, but not composite objects like atoms
    E) All objects

Part II: Short answer [20 points]

Choose **two** of the following questions and give a short answer (2-3 sentences) (10 points each).

11. Before the advent of quantum mechanics, calculations indicated that the energy *E* for each "mode" or wavelength of light could come in any amount from zero to infinity. What assumption did Planck make that modified this formula? At least one formula is required.

Planck had to assume that for each mode, the energy was some multiple of the frequency times a constant, so $E = nhf$, where n can be zero or any positive integer, and h is called the Planck constant. He probably intended to take the limit $h \rightarrow 0$, but found that the formula worked for a finite value of h.

12. The wavelengths produced by hydrogen follow the formula $\lambda = (91.17\text{ nm})\left(\frac{1}{n^2} - \frac{1}{m^2}\right)^{-1}$.

According to the Bohr model, what do *n* and *m* correspond to?

The integers n and m correspond to the final and initial levels of the hydrogen atom respectively. These levels were calculated using the assumption that the electron had to orbit in a circular orbit with angular momentum $L = n\hbar$ for the final level, and $L = m\hbar$ for the initial level.

13. What conclusions about the distribution of positive charges in the atom was Rutherford able to reach by studying how alpha particles scattered off of atoms?

To get scattering, the positive charges had to be concentrated in a tiny area called the nucleus. This nucleus contained much of the atoms mass (actually most of it), and had to be very small. By using higher energy alpha particles, and lower charge nuclei, he was able to demonstrate that this nucleus was only a few fm across.

Part III: Calculation: [60 points]

Choose **three** of the following four questions and perform the indicated calculations (20 points each).

14. An unknown metal is bombarded with light of frequency $f = 1.27\times10^{15}\ \text{s}^{-1}$. It is found that electrons that come out can overcome a potential barrier up to $V_{\max} = 2.05\ \text{V}$.

(a) What is the work function ϕ in eV for this metal?

We use the formula for the photoelectric effect, namely $eV_{\max} = hf - \phi$, which we rearrange to obtain

$$\phi = hf - eV_{\max} = \left(4.136\times10^{-15}\ \text{eV}\cdot\text{s}\right)\left(1.27\times10^{15}\ \text{s}^{-1}\right) - e\left(2.05\ \text{V}\right) = \left(5.25\ \text{eV}\right) - \left(2.05\ \text{eV}\right)$$
$$= 3.20\ \text{eV}.$$

(b) Would electrons continue being emitted, and what would be the maximum potential they could overcome if the light wavelength was changed to $\lambda = 337\ \text{nm}$?

We first need to know the frequency corresponding to this wavelength, which we obtain from $\lambda f = c$, so

$$\lambda f = \frac{c}{\lambda} = \frac{2.998\times10^{8}\ \text{m/s}}{337\times10^{-9}\ \text{m}} = 8.89\times10^{14}\ \text{s}^{-1}.$$

We then can use the same equation before to get the maximum voltage the electrons can overcome, namely

$$eV_{\max} = hf - \phi = \left(4.136\times10^{-15}\ \text{eV}\cdot\text{s}\right)\left(8.89\times10^{14}\ \text{s}^{-1}\right) - \left(3.20\ \text{eV}\right) = \left(3.68\ \text{eV}\right) - \left(3.20\ \text{eV}\right)$$
$$= 0.48\ \text{eV},$$
$$V_{\max} = 0.48\ \text{V}.$$

So electrons can be emitted, but they can only overcome 0.48 V.

(c) What if the light was changed to an angular frequency of $\omega = 4.00\times10^{15}\ \text{s}^{-1}$?

Again, we first need the frequency, which we find from $\omega = 2\pi f$, so

$$f = \frac{\omega}{2\pi} = \frac{4.00\times10^{15}\ \text{s}^{-1}}{2\pi} = 6.37\times10^{14}\ \text{s}^{-1}.$$

If we foolishly simply apply the same formulas as before, we find

$$eV_{\max} = hf - \phi = \left(4.136\times10^{-15}\ \text{eV}\cdot\text{s}\right)\left(6.37\times10^{14}\ \text{s}^{-1}\right) - \left(3.20\ \text{eV}\right) = \left(2.63\ \text{eV}\right) - \left(3.20\ \text{eV}\right)$$
$$= -0.57\ \text{eV}.$$

Since this yields a negative number, we conclude that this light cannot liberate *any* electrons.

15. Alpha particles (*q* = 2*e*) of energy $E = 23.4\text{ MeV}$ and mass $m_\alpha = 3727\text{ MeV}/c^2$ collide with a heavy osmium nucleus (*Z* = 76, *Q* = *Ze*) and are scattered.

(a) For what impact parameter *b* will the alpha particles be scattered by an angle of 112°?

This is a straightforward application of our formula, namely

$$b = \frac{kqQ}{m_\alpha v^2}\cot\left(\frac{\theta}{2}\right) = \frac{k2eZe}{m_\alpha v^2}\cot\left(\frac{\theta}{2}\right) = \frac{kZe^2}{E}\cot\left(\frac{\theta}{2}\right)$$

$$= \frac{\left(8.988\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2\right)76\left(1.602\times10^{-19}\text{ C}\right)^2}{\left(23.4\times10^6\text{ eV}\right)\left(1.602\times10^{-19}\text{ J/eV}\right)\tan\left(\tfrac{1}{2}112°\right)} = 3.15\times10^{-15}\text{ m} = 3.15\text{ fm}.$$

(b) What is the corresponding cross-section σ to be scattered by 112° or more?

Anything that is closer than 3.15 fm will be scattered by more than 112°. The cross-section will be the area of a circle of this radius, which will be

$$\sigma = \pi b^2 = \pi\left(3.15\text{ fm}\right)^2 = 31.3\text{ fm}^2.$$

(c) If the alpha particle were moving directly towards the osmium, what is the closest they will ever come?

This occurs when the kinetic energy *E* gets completely converted into potential energy, $kQq/R = 2kZe^2/R$. Solving for *R*, or simply using the formulas that were given to us (which is easier), we find

$$R = \frac{2kZe^2}{E} = \frac{2\left(8.988\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2\right)76\left(1.602\times10^{-19}\text{ C}\right)^2}{\left(23.4\times10^6\text{ eV}\right)\left(1.602\times10^{-19}\text{ J/eV}\right)} = 9.35\times10^{-15}\text{ m} = 9.35\text{ fm}.$$

(d) If we wanted the alpha particle instead to get within 1 fm of the nucleus, to what value of *Z* would we have to decrease the target?

We can see from the formula in the previous part that the radius reached is proportional to *Z*, so to decrease it to *R* = 9.35 fm, we would have to decrease *Z* by a factor of 9.35. So we need

$$Z' < \frac{Z}{9.35} = \frac{76}{9.35} = 8.13.$$

Since *Z* has to be an integer, this means it is 8 or less, so elements up through oxygen.

16. Ant Man needs to go inside a diamond crystal, and as such he needs to squeeze between two layers of atoms only about 1.725×10^{-10} m apart.

(a) Use Carlson's rule to estimate the uncertainty in the position Δx.

Carlson's rule says that the uncertainty is one-fourth of the separation, so

$$\Delta x = 0.250\left(1.725\times10^{-10}\ \text{m}\right) = 4.312\times10^{-11}\ \text{m}.$$

(b) What is the corresponding minimum uncertainty in his momentum Δp in kg/m/s?

We use the uncertainty relation $(\Delta x)(\Delta p) \geq \frac{1}{2}\hbar$ to find

$$\Delta p \geq \frac{\hbar}{2(\Delta x)} = \frac{1.055\times10^{-34}\ \text{J}\cdot\text{s}}{2\left(4.312\times10^{-11}\ \text{m}\right)} = 1.223\times10^{-24}\ \text{J}\cdot\text{s/m} = 1.223\times10^{-24}\,\text{kg}\cdot\text{m/s}.$$

For a macroscopic object (like a person) this is a tiny uncertainty, but for something like an electron, this would be very small.

(c) If Ant Man, when shrunk, still has his ordinary mass of 86.0 kg, what would be the corresponding energy in J?

The kinetic energy is just $E = \frac{1}{2}mv^2$, and the velocity is given by $v = p/m$. If we treat the uncertainty in the momentum as if it were the momentum, we would have

$$E = \frac{mv^2}{2} = \frac{p^2}{2m} \approx \frac{(\Delta p)^2}{2m} = \frac{\left(1.223\times10^{-24}\ \text{kg}\cdot\text{m/s}\right)^2}{2(86.0\ \text{kg})} = 8.70\times10^{-51}\ \text{J}.$$

17. The wave function for a particle is given by $\psi(x) = Nxe^{-Ax^2}$ where A is a positive constant. A possibly helpful integral is given below.

(a) Where is it impossible for the particle to be? Where is it most likely to be?

The particle can not be at places where the wave function vanishes, which is when $0 = \psi(x) = Nxe^{-Ax^2}$, which happens only at $x = 0$. The particle is most likely to be at places where the square of the magnitude of the wave function is maximized, which for a real function, is anywhere the wave function has a maximum or minimum. These can be found by setting the derivative to zero, so we have

$$0 = \frac{d}{dx}\psi(x) = N\frac{d}{dx}\left(xe^{-Ax^2}\right) = N\left[e^{-Ax^2} - x(2Ax)e^{-Ax^2}\right] = Ne^{-Ax^2}\left(1-2Ax^2\right),$$

$$2Ax^2 = 1,$$

$$x = \pm\frac{1}{\sqrt{2A}}.$$

(b) What is the normalization constant N?

We apply the normalization condition, which says that

$$1 = \int_{-\infty}^{\infty}\left|\psi(x)\right|^2 dx = \int_{-\infty}^{\infty}\left(Nxe^{-Ax^2}\right)^2 dx = N^2\int_{-\infty}^{\infty} x^2 e^{-2Ax^2} dx = N^2\frac{\Gamma\left(\frac{3}{2}\right)}{(2A)^{3/2}} = N^2\frac{\frac{1}{2}\sqrt{\pi}}{2A\sqrt{2A}},$$

$$N = \sqrt{\frac{4A\sqrt{2A}}{\sqrt{\pi}}} = 2\left(\frac{2A^3}{\pi}\right)^{1/4}.$$

It's messy, but it is what it is.

(c) What is the probability density that the particle is at the points $x = \pm 1/\sqrt{A}$?

We simply substitute in this expression into the square of the wave function, so we have

$$\left|\psi\left(\pm\tfrac{1}{\sqrt{A}}\right)\right|^2 = N^2\left(\pm\tfrac{1}{\sqrt{A}}\right)^2\exp\left[-2A\left(\pm\tfrac{1}{\sqrt{A}}\right)^2\right] = \frac{N^2}{A}\exp(-2) = \frac{4}{A}\sqrt{\frac{2A^3}{\pi}}e^{-2} = \frac{4\sqrt{2A}}{e^2\sqrt{\pi}}.$$

Possibly helfpul integral:

$$\int_{-\infty}^{\infty} x^n e^{-Bx^2} dx = \begin{cases}\Gamma\left(\frac{n+1}{2}\right)\Big/B^{(n+1)/2} & n \text{ even,}\\ 0 & n \text{ odd,}\end{cases} \quad \text{where} \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}.$$