Solutions to Test 2
October 13, 2021

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered. The equations below may be helpful with some problems.

Constants

$h = 6.626\times10^{-34}\ \text{J}\cdot\text{s}$

$h = 4.136\times10^{-15}\ \text{eV}\cdot\text{s}$

$\hbar = 1.055\times10^{-34}\ \text{J}\cdot\text{s}$

$\hbar = 6.582\times10^{-16}\ \text{eV}\cdot\text{s}$

$k_B = 1.3807\times10^{-23}\ \text{J/K}$

$k_B = 8.6173\times10^{-5}\ \text{eV/K}$

$k = 8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2$

$e = 1.602\times10^{-19}\ \text{C}$

$1\ \text{eV} = 1.602\times10^{-19}\ \text{J}$

$\alpha = \dfrac{ke^2}{\hbar c} \simeq \dfrac{1}{137}$

Hydrogen-Like Atoms

$$E = -\frac{k^2e^4\mu Z^2}{2\hbar^2 n^2} = -\frac{(\mu c^2)\alpha^2 Z^2}{2n^2}$$

$$E = \frac{-(13.60\ \text{eV})Z^2}{n^2}$$

Hydrogen Spectrum

$$\lambda = (91.17\ \text{nm})\left(\frac{1}{n^2}-\frac{1}{m^2}\right)^{-1}$$

Black Bodies

$$U = \frac{\pi^2(k_BT)^4}{15(\hbar c)^3}$$

$$\lambda_{\text{max}}T = .002898\ \text{m}\cdot\text{K}$$

Wave Relationships

$$\lambda = \frac{2\pi}{k}$$

$$\frac{\omega}{2\pi} = f = \frac{1}{T}$$

Reduced Mass

$$\mu = \frac{mM}{m+M}$$

Compton Effect

$$\lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$$

$$\frac{h}{mc} = 2.426\times10^{-12}\ \text{m}$$

Rutherford Scattering

$$b = \frac{kqQ}{m_\alpha v^2}\cot\left(\frac{\theta}{2}\right)$$

$$R = \frac{2Ze^2k}{E}$$

Part I: Multiple Choice [20 points]

For each question, choose the best answer (2 points each).

1. The frequency of a photon with energy E is given by

   A) E/h B) h/E C) hE D) $E/\hbar$ E) $\hbar/E$

2. If a wave function looks like $\psi = a+bi$, where a and b are real numbers, which of the following would be the probability density?

   A) $a+b$ B) $(a+bi)^2$ C) $\sqrt{a^2+b^2}$ **D)** a^2+b^2 E) $a-bi$

3. If you have an electron (mass m) and an anti-electron (also mass m) circling each other, what would be the reduced mass μ for the two objects?

   A) $2m$ **B)** $\dfrac{m}{2}$ C) $\dfrac{2}{m}$ D) $\dfrac{1}{2m}$ E) $\dfrac{1}{m}$

4. By studying the motion of tiny oil drops in a strong electric field, Millikan was able to
   A) Find the mass of the electron
   B) Demonstrate that electrons are negatively charged
   C) Measure the velocity to mass ratio of the electrons
   D) Show that the mass and positive charge of an atom was concentrated in the nucleus
   E) Demonstrate (and measure) that charges are integer multiples of the fundamental charge *e*

5. The Franck-Hertz experiment, which demonstrated that electrons collide elastically (no energy loss) with atoms until they reach a certain threshold of energy, demonstrated that which of the predictions of the Bohr model was correct?
   A) Atoms have discrete energy levels
   B) Electrons actually do orbit in circular orbits
   C) The angular momentum is always an integer multiple of $\hbar$
   D) The size of an atom is proportional to n^2, where n is the level number
   E) Atoms have wave-like properties

6. The Bohr model of the atom predicts a hydrogen atom is about 0.1 nm in diameter. How does this compare with the actual size of a hydrogen atom?
   A) It is too large by about a factor of 100 or more
   B) It is too large by about a factor of 10
   C) It is about right
   D) It is too small by about a factor of 10
   E) It is too small by about a factor of 100

7. For hydrogen-like atoms, you can get the energy of all states from the formula $E = -(13.60 \text{ eV})Z^2/n^2$. For complicated atoms with many electrons, this formula almost works for the lowest levels (n = 1 or 2), but you have to modify it slightly. What modification is necessary?
   A) You have to increase the n value because the lower level n values have all been filled
   B) You have to take into account the reduced mass formula because the nucleus moves a lot
   C) You have to reduce the *Z* value, because the nuclear charge *Ze* is screened a little bit by the other electrons
   D) The value of n must be multiplied by Z to take into account the increased nuclear charge
   E) None; the formula still works quite well without modification

8. Which of the following is the normalization condition for a wave function for a particle in one dimension $\psi(x)$?
   A) $|\psi| = 1$ B) $|\psi|^2 = 1$ C) $\int \psi \, dx = 1$ D) $\int \psi^2 dx = 1$ **E)** $\int |\psi|^2 \, dx = 1$

9. If you want to actually measure the size of a nucleus by scattering α-particles off of atoms, it helps to make the energy E _______ and the nuclear charge Ze ______ .
   A) Small, large B) Large, large C) Small, small **D) Large, small** E) Even, odd

10. Waves are most likely to diffract when they pass through a slit if
    A) The wavelength is smaller than the slit size
    B) The wavelength is bigger than the slit size
    C) The group velocity is greater than the phase velocity
    D) The group velocity is smaller than the phase velocity
    E) Waves do not diffract; this is a particle property, not a wave property

Part II: Short essay [20 points]

Choose two of the following three questions and write a short essay (2-3 sentences). Each question is worth 10 points.

11. When you shine a light on a metal, sometimes electrons pop off, and sometimes they do not. Explain qualitatively why this happens, and also explain how the energy of the ejected electrons depends on the frequency of the light emitted. A formula would be a good idea.

It takes a certain amount of energy to remove an electron from a metal. The minimum is called the work function, denoted ϕ. The photons each have energy hf, and therefore they are only capable of removing electrons if $hf > \phi$. When this is the case, the energy of the electron can have a maximum energy of $E_{\max} = hf - \phi$. If it is then required to overcome an electric potential, the maximum potential it can overcome is $eV_{\max} = hf - \phi$.

12. Explain qualitatively what the difference between the group velocity and phase velocity of a wave is. Also, give formulas for them if we have the angular frequency ω as a function of wave number, *k*.

The phase velocity is just the rate at which individual waves travel within the packet, and the group velocity is the rate at which the wave packet travels. If we have a formula for ω, then the phase and group velocity are given by $v_p = \frac{\omega}{k}$ and $v_g = \frac{d\omega}{dk}$ respectively.

13. The harmonic oscillator will have zero potential energy if the particle is at the center *x* = 0, and it will have zero kinetic energy if it has no momentum, *p* = 0. Hence the minimum energy of the harmonic oscillator is zero. Explain from a quantum mechanical viewpoint what is wrong with this argument.

According to quantum mechanics, a particle's position is described by a wave function which doesn't have a specific position or momentum. In particular, the wave function will satisfy the uncertainty relation $(\Delta x)(\Delta p) \geq \frac{1}{2}\hbar$. Hence there will be a minimum zero-point energy for the harmonic oscillator.

Part III: Calculation: [60 points]
Choose three of the following four questions. Each question is worth 20 points.

14. At a time called "matter-radiation equality," the universe was filled with electromagnetic black body radiation with an energy density of $U = 5.56\ \text{J/m}^3$.

a) What was the temperature in Kelvin at this time?

We use the formula for the energy density as a function of temperature, and solve for the temperature:

$$U = \frac{\pi^2 (k_B T)^4}{15(\hbar c)^3},$$

$$(k_B T)^4 = \frac{15}{\pi^2} U (\hbar c)^3 = \frac{15}{\pi^2}\left(5.56\ \text{J/m}^3\right)\left[\left(1.055\times10^{-34}\ \text{J}\cdot\text{s}\right)\left(2.998\times10^{8}\ \text{m/s}\right)\right]^3 = 2.670\times10^{-76}\text{J}^4,$$

$$k_B T = \left(2.670\times10^{-77}\text{J}^4\right)^{1/4} = 1.278\times10^{-19}\ \text{J},$$

$$T = \frac{1.278\times10^{-19}\ \text{J}}{k_B} = \frac{1.278\times10^{-19}\ \text{J}}{1.381\times10^{-23}\ \text{J/K}} = 9260\ \text{K}.$$

b) What was the wavelength at which this thermal spectrum peaked at this time?

We can deduce this from the formula $\lambda_{\max} T = .002898\ \text{m}\cdot\text{K}$, from which we can deduce

$$\lambda_{\max} = \frac{0.002898\ \text{m}\cdot\text{K}}{T} = 3.13\times10^{-7}\ \text{m} = 313\ \text{nm}.$$

c) What was the energy of a single photon, in eV, at the wavelength found in part (b)?

The first step is to get the frequency, which we can deduce from $f\lambda = c$, and then we can get the energy from $E = hf$. So we have

$$E = hf = h\left(\frac{c}{\lambda}\right) = \frac{\left(4.136\times10^{-15}\ \text{eV}\cdot\text{s}\right)\left(2.998\times10^{8}\ \text{m/s}\right)}{3.13\times10^{-7}\ \text{m}} = 3.96\ \text{eV}.$$

15. An X-ray scatters from electrons at rest. It is found for those X-rays scattering at a 90° angle, the scattered photons have a wavelength of $\lambda' = 8.98\text{ pm} = 8.98\times10^{-12}\text{ m}$.

a) What is the wavelength of the incoming X-rays?

This problem repeatedly uses the formula for the Compton effect, $\lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$, where $h/mc = 2.426\text{ pm}$. For part (a), we solve this for λ, which gives

$$\lambda = \lambda' - \frac{h}{mc}(1-\cos\theta) = (8.98\text{ pm}) - (2.426\text{ pm})(1-\cos(90°)) = (8.98\text{ pm}) - (2.426\text{ pm})$$
$$= 6.554\text{ pm}.$$

We will then use this in subsequent formulas.

b) What would be the wavelength of X-rays scattered at 180°?

We have

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta) = (6.554\text{ pm}) + (2.426\text{ pm})(1-\cos(180°))$$
$$= (6.554\text{ pm}) + (2.426\text{ pm})\times 2 = 11.41\text{ pm}.$$

c) At what angle θ would the scattered wavelength be $\lambda' = 8.26\text{ pm} = 8.26\times10^{-12}\text{ m}$**?**

This time we solve for the angle, so we have

$$\frac{h}{mc}(1-\cos\theta) = \lambda' - \lambda,$$
$$(2.426\text{ pm})(1-\cos\theta) = (8.26\text{ pm}) - (6.554\text{ pm}) = 1.706\text{ pm},$$
$$1-\cos\theta = \frac{1.706\text{ pm}}{2.426\text{ pm}} = 0.703,$$
$$\cos\theta = 1 - 0.703 = 0.297,$$
$$\theta = \cos^{-1}(0.297) = 72.7°.$$

16. A silicon (Z = 14) atom has a single electron in level n = 8. The electron then shifts to n = 7.

a) What are the energies of the initial and final states?

The energies are just given by $E_n = -(13.60 \text{ eV}) Z^2 / n^2$. We therefore have

$$E_8 = -\frac{(13.60 \text{ eV})14^2}{8^2} = -41.65 \text{ eV},$$
$$E_7 = -\frac{(13.60 \text{ eV})14^2}{7^2} = -54.40 \text{ eV}.$$

b) When the atom transitions, would it emit or absorb a photon? Find the energy of the corresponding photon.

The energy is becoming more negative, so it lost energy. This implies that the atom must be emitting a photon. The energy of the photon is the initial minus final energy, so

$$\Delta E = E_8 - E_7 = (-41.65 \text{ eV}) - (54.40 \text{ eV}) = 12.75 \text{ eV}.$$

c) A nearby hydrogen atom in the ground state (n = 1) absorbs a photon with the energy from part (b). What level n will this atom end up in?

Hydrogen has Z = 1, so the energy of the n'th level is simply $E_n = -(13.6 \text{ eV})/n^2$, or for the ground state, $E_n = -13.6 \text{ eV}$. The final energy is then this initial energy plus the change, so

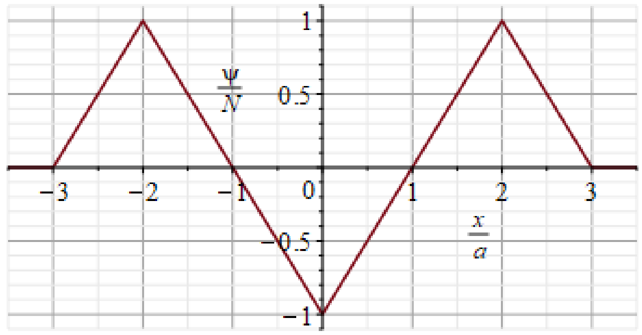
$$E_f = (-13.6 \text{ eV}) + (12.75 \text{ eV}) = -0.85 \text{ eV}.$$

We now simply equate this to the general formula $E_n = -(13.6 \text{ eV})/n^2$ and solve for n:

$$E_n = \frac{-13.60 \text{ eV}}{n^2} = -0.85 \text{ eV},$$
$$n^2 = \frac{13.60 \text{ eV}}{0.85 \text{ eV}} = 16.0,$$
$$n = \sqrt{16.0} = 4.00.$$

Of course, n is an integer, but we were lucky that rounding it came out with the value 4.00, and the correct answer is just n = 4.

17. A particle in the range $-3a < x < 3a$ has a wave function as sketched at right. Note that the horizontal axis is labeled in terms of *x*/*a*, so that, for example, the value 0.5 corresponds to *x* = 0.5*a*.



a) In the allowed region, where is it impossible for the particle to be?

The particle cannot be any place that the wave function vanishes. Looking at the sketch, this is at $x/a = \pm 1$, or $x = \pm a$. It also vanishes at the endpoints $x = \pm 3a$, but this is at the boundary and hence not in our allowed region.

b) In the allowed region, where is the particle most likely to be?

The particle is most likely to be at any point where the wave function is most positive or most negative. It is clear that $\psi/N = +1$ at $x = \pm 2a$, while $\psi/N = -1$ at $x = 0$. Since all that matters is the square of the wave function, the three points $\{-2a, 0, 2a\}$ are all equally likely.

c) In the region $0 < x < a$, the wave function is given by $\psi(x) = \dfrac{x-a}{\sqrt{2a^3}}$. What is the probability the particle is in this region?

We simply integrate the amplitude squared over the preferred region, so we have

$$P(0 < x < a) = \int_0^a |\psi(x)|^2\,dx = \frac{1}{2a^3}\int_0^a (x-a)^2\,dx = \frac{1}{2a^3}\int_0^a \left(x^2 - 2ax + a^2\right)dx$$

$$= \frac{1}{2a^3}\left(\frac{1}{3}x^3 - ax^2 + a^2 x\right)\Bigg|_0^a = \frac{1}{2a^3}\left[\left(\frac{1}{3}a^3 - a^3 + a^3\right) - (0)\right] = \frac{1}{2a^3}\cdot\frac{1}{3}a^3 = \frac{1}{6} = 16.7\%.$$

d) Based on the fact that the particle is in the region $-3a < x < 3a$, estimate the position uncertainty Δx (you can use Carlson's rule). Use this to get an approximate lower limit on the momentum uncertainty Δp.

The region has a width of $6a$. Carlson's rule says that the uncertainty in the position is one-fourth of this width, so

$$\Delta x \approx \tfrac{1}{4}\cdot 6a = \tfrac{3}{2}a.$$

The Heisenberg uncertainty principle tells us that $(\Delta x)(\Delta p) \geq \tfrac{1}{2}\hbar$. We therefore have

$$\Delta p \geq \frac{\hbar}{2(\Delta x)} \approx \frac{\hbar}{2\left(\frac{3}{2}a\right)} = \frac{\hbar}{3a}.$$