Solution to Test 4
November 4, 2020

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered.

Part I: Multiple Choice [20 points]
For each question, choose the best answer (2 points each). Your test will have had these questions in a random order.

2. The solutions of Schrodinger's time-dependent equation can take the form $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$ what conditions?
   A) If the times are sufficiently small
   B) If the potential is independent of time
   C) If the potential is continuous
   D) If the wave function is normalized
   E) If the potential is symmetric, so that $V(x,t) = V(-x,t)$

3. If you take a single photon and send it through a partially silvered mirror which divides the wave function in two, and then measure which way that single photon went, what would happen, assuming perfect detectors?
   A) You get two photons, each one with half the energy of the original
   B) 50% of the time you get it in one detector, 50% in the other, but never neither and never both
   C) 25% in both detectors, 25% in neither, 25% in one of them, and 25% in the other one
   D) 50% of the time in both detectors, and 50% in neither
   E) Nothing happens until you put two photons in, in which case you get one in each

4. Suppose you are solving Schrodinger's equation, and there is a sudden (but finite) change in the potential. What sort of condition do you want at the boundary between the two regions?
   A) Match the wave function (only)
   B) Match the derivative of the wave function (only)
   C) Match the wave function AND its derivative
   D) The wave function must vanish
   E) The derivative of the wave function must vanish

5. If a particle has an energy E and impacts a thick barrier of width L and height $V_0 > E$, what will happen?
   A) It will be 100% reflected
   B) It will be 100% transmitted
   C) It will mostly be reflected, with a tiny bit of transmission
   D) It will be mostly transmitted, with a tiny bit of reflection
   E) It will be both transmitted and reflected, with comparable probabilities

6. If Ψ_1 and Ψ_2 are solutions of Schrodinger's equation, which of the following must also be a solution?
   A) $\Psi_1 + \Psi_2^*$ B) $\Psi_1 \cdot \Psi_2$ C) Ψ_1 / Ψ_2 D) $3\Psi_1^* + 4\Psi_2^*$ **E)** $\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_2)$

7. What sort of behavior would you generally want for a solution to Schrodinger's equation if it is going to be normalizable?
   A) It should have a continuous derivative
   B) It should go to a constant at infinity
   C) It should vanish at at least one place
   D) It should fall off to zero at infinity
   E) It should be continuous

8. Which of the following happens when you take quantum mechanics to three dimensions?
   A) The wave function becomes a function of all three coordinates, $\Psi(x, y, z, t)$
   B) The wave function becomes a sum of wave functions for each of the three dimensions
   C) Schrodinger's equation becomes three separate equations, one for each dimension
   D) Probabilities are no longer described by integrals of $|\Psi|^2$
   E) You must make the wave function anti-symmetric when you interchange the various space coordinates *x*, *y,* and *z*.

9. Bound states occur if
   A) The wave function is exactly zero outside of a small region
   B) Real wave functions satisfy Schrodinger's equations
   C) The potential is larger than the energy at at least one point in the interior
   D) The energy is more than the potential at infinity
   E) The energy is less than the potential at infinity

10. When you calculate an expectation value of the position operator $\langle x \rangle$, what does it tell you?
    A) The most likely place to find the particle
    B) The place the particle will be, but only if it is measured
    C) The place to which the wave function will be attracted over time
    D) The average value if you were to measure the position
    E) The largest or smallest value of the position you could measure

11. What would be an appropriate complete set of initial conditions for a wave function governed by Schrodinger's equation?
    A) The initial wave function $\Psi(x, t = 0)$
    B) The initial wave function $\Psi(x, t = 0)$ and its first time derivative $\dot{\Psi}(x, t = 0)$
    C) The initial wave function $\Psi(x, t = 0)$ and the initial position of the particle $x(t = 0)$
    D) The initial positon $x(t = 0)$ and momentum $p(t = 0)$ of the particle
    E) Just the initial position $x(t = 0)$

Part II: Short essay [20 points]

Choose two of the following three questions, and write a short essay (2-3 sentences). You may type both answers into the answer box at the end, or you may upload your answers as an image into the box. Each question is worth 10 points.

12A. For what sort of potential do the energy values come out evenly spaced? Find a formula, in this case, for the difference in energy between level *n* and level *m*.

The harmonic oscillator has energy values of $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$. These are evenly spaced, so the answer is the harmonic oscillator. The difference between level *n* and level *m* is $E_n - E_m = \hbar\omega\left(n-m\right)$.

12B. Explain what the expectation value of the Hamiltonian $\langle H \rangle$ corresponds to if we were to measure it for a given wave function.

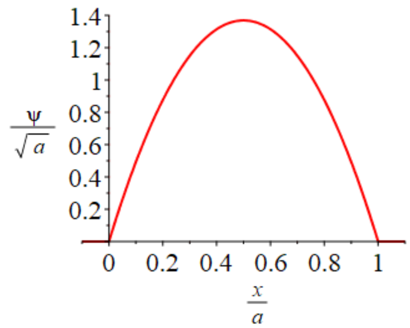
The expectation value of the Hamiltonian is the average value if you were to measure the energy of a system with a given wave function.

12C. For the hydrogen atom, we wrote the wave function in the form $\psi = R\left(r\right)Y\left(\theta,\phi\right)$. Why does this make sense for the hydrogen atom, but not, for example, for the 3D box?

The hydrogen atom has spherically symmetric, and therefore it make sense to factor the wave function into a function of the radius times functions of angles. This approach makes no sense for the 3D box.

Part III: Calculation: [60 points]

Choose three of the following four questions. Each question is worth 20 points. Type only your answers to each part into the essay box provided.



13. A wave function in the region $0 < x < a$ is given by $\psi(x) = \sqrt{30/a^5}\left(ax - x^2\right)$, as sketched at right. Outside this region, the wave function vanishes. This wave function has expectation values $\langle x \rangle = \frac{1}{2}a$ and $\langle x^2 \rangle = \frac{2}{7}a^2$.

a) What are the expectation values of $\langle p \rangle$ and $\langle p^2 \rangle$?

Because the wave function is real, we automatically have $\langle p \rangle = 0$. For $\langle p^2 \rangle$, we have

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* p_{op}^2 \psi \, dx = \frac{30}{a^5}\int_0^a \left(ax - x^2\right)\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 \left(ax - x^2\right) dx = \frac{-30\hbar^2}{a^5}\int_0^a \left(ax - x^2\right)\frac{d^2}{dx^2}\left(ax - x^2\right) dx$$

$$= \frac{-30\hbar^2}{a^5}\int_0^a \left(ax - x^2\right)(-2)\, dx = \frac{60\hbar^2}{a^5}\left(\tfrac{1}{2}ax^2 - \tfrac{1}{3}x^3\right)\Big|_0^a = \frac{60\hbar^2 a^3}{a^5}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{10\hbar^2}{a^2}.$$

b) What are the uncertainties Δx and Δp?

We calculate these in the usual way:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{2}{7}a^2 - \left(\frac{1}{2}a\right)^2} = a\sqrt{\frac{2}{7} - \frac{1}{4}} = \frac{a}{\sqrt{28}},$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{10\hbar^2}{a^2} - 0^2} = \frac{\hbar\sqrt{10}}{a}.$$

c) Does this wave function satisfy the uncertainty relation?

It must, but let's check it:

$$(\Delta x)(\Delta p) = \frac{a}{\sqrt{28}} \cdot \frac{\hbar\sqrt{10}}{a} = \hbar\sqrt{\frac{5}{14}} \approx 0.598\hbar \geq \tfrac{1}{2}\hbar.$$

14. A group of 3,600 particles have energy E = 12 eV. They approach a step potential of size V_0. It is found that 3,200 of them are transmitted.

a) What is the reflection probability R?

If 3,200 of them are transmitted out of 3,600, then 400 of them are reflected, so the reflection probability is

$$R = \frac{400}{3,600} = \frac{1}{9} \approx 0.111 .$$

b) What are the possible values of V_0?

We set this reflection probability equal to the formula, and we have

$$\frac{1}{9} = R = \left(\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}} \right)^2 ,$$

$$\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}} = \pm\sqrt{\frac{1}{9}} = \pm\frac{1}{3} ,$$

$$3\left(\sqrt{E} - \sqrt{E - V_0}\right) = \pm\left(\sqrt{E} + \sqrt{E - V_0}\right)$$

We now split it into two cases and then simplify. For the plus case we have

$$3\sqrt{E} - 3\sqrt{E - V_0} = \sqrt{E} + \sqrt{E - V_0} ,$$

$$2\sqrt{E} = 4\sqrt{E - V_0} ,$$

$$E = 4\left(E - V_0\right),$$

$$4V_0 = 3E ,$$

$$V_0 = \tfrac{3}{4}E = \tfrac{3}{4}\left(12 \text{ eV}\right) = 9 \text{ eV}.$$

For the minus case we have

$$3\sqrt{E} - 3\sqrt{E - V_0} = -\sqrt{E} - \sqrt{E - V_0} ,$$

$$4\sqrt{E} = 2\sqrt{E - V_0} ,$$

$$4E = E - V_0,$$

$$3E = -V_0 ,$$

$$V_0 = -3E = -3\left(12 \text{ eV}\right) = -36 \text{ eV}.$$

Hence the two answers are 9.0 eV or –36 eV.

15. An electron ($m = 9.11\times10^{-31}$ kg) in a one-dimensional box is in the *n* = 5 state. When it falls from this state to the lowest energy state, it emits a photon with an energy of $E = 2.31\times10^{-19}$ J.

a) Find a formula for the difference in energy between the two states in terms of *m* and *L*.

The energy for a particle of mass *m* in a 1D box of size *L* is $E_n = \pi^2\hbar^2 n^2 / \left(2mL^2\right)$. Therefore the difference between the *n* = 5 state and the ground state (*n* = 1) is

$$\Delta E = E_5 - E_1 = \frac{\pi^2\hbar^2 5^2}{2mL^2} - \frac{\pi^2\hbar^2 1^2}{2mL^2} = \frac{\pi^2\hbar^2 24}{2mL^2} = \frac{12\pi^2\hbar^2}{mL^2}.$$

b) What is the length *L* of this box in nm?

The difference in energy is the same as the energy released by the photon. Solving for *L*, we have

$$L^2 = \frac{12\pi^2\hbar^2}{m\Delta E} = \frac{12\pi^2\left(1.0546\times10^{-34}\ \text{J}\cdot\text{s}\right)^2}{\left(9.11\times10^{-31}\ \text{kg}\right)\left(2.31\times10^{-19}\ \text{J}\right)} = 6.259\times10^{-18}\,\text{J}\cdot\text{s}^2/\text{kg} = 6.259\times10^{-18}\ \text{m}^2,$$

$$L = \sqrt{6.259\times10^{-18}\,\text{m}^2} = 2.502\times10^{-9}\ \text{m} = 2.50\ \text{nm}.$$

c) Suppose a photon with the same energy had been absorbed, instead of emitted. What level would it be in now?

It is easier to work with formulas. The new energy E_n will be the sum of E_5 and the difference in energy, so we have

$$E_n = E_5 + \Delta E = \frac{\pi^2\hbar^2 5^2}{2mL^2} + \frac{12\pi^2\hbar^2}{mL^2} = \frac{\pi^2\hbar^2\left(25+24\right)}{2mL^2} = \frac{49\pi^2\hbar^2}{2mL^2}.$$

Matching this to the general equation, we have

$$E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2} = \frac{49\pi^2\hbar^2}{2mL^2},$$

$$n^2 = 49,$$

$$n = 7.$$

So it ended up in the seventh level.

16. A single electron in a hydrogen atom has its total angular momentum squared and it is given by $L^2 = 12\hbar^2$. Recall that the energy of a hydrogen atom is given by $E = -(13.6 \text{ eV})/n^2$.

a) What is the value of the quantum number *l*?

Keeping in mind the formula $L^2 = \hbar^2 (l^2 + l)$, we have $l^2 + l = 12$ and remembering that l is a non-negative integer, the only solution to this equation is $l = 3$.

For each of the quantities below, tell how many possible values there could be, and what the smallest (most negative) value it could be. Some answers may be infinite.

b) The angular momentum around the *z*-axis L_z

The angular momentum around the *z*-axis is given by $L_z = \hbar m$, where m runs from $-l$ to l, so $m = -3, -2, -1, 0, 1, 2, 3$, for a total of **seven** values. The most negative of these is -3, so that would be $L_z = -3\hbar$.

c) The spin around the *z*-axis S_z

The spin of an electron only takes on values of $S_z = \hbar m_s$, where $m_s = \pm\frac{1}{2}$ so there are **two** values. The most negative would be $S_z = -\frac{1}{2}\hbar$.

d) The energy *E*

Remembering that l takes on values from 0 to $n-1$, it is clear that n must be larger than l. For $l = 3$, this implies $n \geq 4$, but there are infinity possible results. The lowest energy state would be the one with the smallest n, $n = 4$, so that $E = -(13.6 \text{ eV})/4^2 = -0.850 \text{ eV}$.