Solutions to Test 3
November 3, 2021

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered. Some possibly useful formulas can be found below.

$$h = 6.626\times10^{-34}\ \text{J}\cdot\text{s} = 4.136\times10^{-15}\ \text{eV}\cdot\text{s}$$
$$\hbar = 1.055\times10^{-34}\ \text{J}\cdot\text{s} = 6.582\times10^{-16}\ \text{eV}\cdot\text{s}$$
$$1\ \text{eV} = 1.602\times10^{-19}\ \text{J}$$

Barrier penetration:
$$T \approx 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\alpha L}$$
$$\alpha = \sqrt{2m(V_0-E)}\Big/\hbar$$

1D square well:
$$E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2}$$
$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi n x}{L}\right)$$
$$n = 1,2,3,\ldots$$

Reflection off a step:
$$R = \begin{cases} \left(\dfrac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}}\right)^2 & \text{if } E > V_0 \\ 1 & \text{if } E < V_0 \end{cases}$$

Harmonic Oscillator
$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$
$$n = 0,1,2,\ldots$$

Hydrogen
$$E_n = -\frac{(13.6\ \text{eV})Z^2}{n^2}$$

Part I: Multiple Choice [20 points]
For each question, choose the best answer (2 points each)

1. Suppose that a single photon is sent through a half-silvered mirror, so that the wave function is reflected 50% of the time and transmitted 50% of the time. We then place detectors to see which way it goes. Assuming the detectors are perfect, what is the probability that *neither* of them will see the photon?
   A) 0% B) 25% C) 50% D) 100% E) None of these

2. For which of the following problems is the energy spacing between level *n* and level *n*+1 always the same, independent of *n*?
   A) Hydrogen atom
   B) Infinite square well
   C) Finite square well
   D) Harmonic oscillator
   E) None of the above

3. Suppose you solve Schrodinger’s equation in two regions. What constraint(s) should you impose at the boundary between the two regions?
   A) Match the wave function (only)
   B) Match the derivative of the wave function (only)
   C) Match BOTH the wave function and the derivative
   D) Make sure both wave functions vanish at the boundary
   E) Make sure both wave functions are finite at the boundary

4. Which of the following might be a good statement of the Pauli exclusion principle for electrons?
   A) Electrons repel each other, so they tend not to be near each other
   B) If one electron is in the *n*'th level of a hydrogen atom, no other electron can be in that level
   C) Some levels of hydrogen-like atoms are excluded from containing an electron
   D) Electrons are excluded from the nucleus, because it is already filled with protons and neutrons
   E) Two electrons can never be in the same quantum state

5. If you have a 4d electron, what are the values of *n* and *l*?
   A) $n = 3,\ l = 4$ B) $n = 4,\ l = 3$ C) $n = 2,\ l = 4$ **D)** $n = 4,\ l = 2$ E) None of these

6. Suppose you have a solution to Schrodinger's equation, but it is not properly normalized, so you find $\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 4$. What should you multiply $\psi(x)$ by to fix this?
   A) $\frac{1}{4}$ **B)** $\frac{1}{2}$ C) $\frac{1}{16}$ D) 2 E) 4

7. Which of the following might represent the wave function for a particle moving in the $+x$ direction?
   A) e^{-ikx} **B)** e^{ikx} C) $\cos(kx)$ D) $\sin(kx)$ E) e^{-Ax^2}

8. Which of the following is the momentum operator for a particle in one dimension?
   A) $\frac{\hbar}{i}\frac{d}{dx}$ B) $i\hbar\frac{d}{dx}$ C) $i\hbar x$ D) $-i\hbar x$ E) None of these

9. When we look at hydrogen wave functions, for which types of wave functions have a chance that the electron is near the origin?
   A) Those with small values of *n*
   B) Those with large values of *n*
   C) Those with small values of *l*
   D) Those with large values of *l*
   E) The electron can never be near the origin for any hydrogen wave function

10. When does the uncertainty relation $(\Delta x)(\Delta p) \geq \frac{1}{2}\hbar$ **not** apply?
   A) Bound states
   B) Unbound states
   C) When the energy is positive
   D) When the energy is negative
   E) Never, it always applies

Part II: Short answer [20 points]

Choose two of the following three questions and give a short answer (2-4 sentences) (10 points each).

11. What sorts of potentials $V(\mathbf{r},t)$ in three dimensions have the property such that it makes sense to write solutions like $\Psi(\mathbf{r},t)=\psi(\mathbf{r})e^{-iEt/\hbar}$, factoring it into a time component and a space component? For which sorts of potentials does it make more sense to write the space part in spherical coordinates as $\psi(\mathbf{r})=R(r)Y_{l,m}(\theta,\phi)$?

The wave function can be factored into a time and space part whenever the potential is independent of time, so $V(\mathbf{r},t)=V(\mathbf{r})$. The wave function *should* be solved in spherical coordinates, and in fact will factor into a radial part and an angular part, whenever the potential depends only on the distance from the origin, so $V(\mathbf{r})=V(r)$.

12. For a particle in one dimension with mass *m* and potential $V(x)$, what is the formula for the Hamiltonian? If I take a wave function $\psi(x)$ and use it to calculate the expectation value of the Hamiltonian $\langle H\rangle$, what would this correspond to?

The Hamiltonian is given by $H = p_{op}^2/2m+V(x_{op})$, where we have added the "op" subscript to remind as that these should be interpreted as operators, so for example $p_{op}^2=-\hbar^2\dfrac{d^2}{dx^2}$. The expectation value of the Hamiltonian is the average energy you would measure if you measure the energy of this wave function.

13. A particle of energy *E* impacts a step potential of size V_0. Explain if the wave is partly reflected, completely reflected, or completely transmitted in the four cases $E<V_0$, $0<V_0<E$, $V_0=0$ and $V_0<0$. No equations are necessary.

The wave will be partly reflected and partly transmitted both in the cases $0<V_0<E$ and $V_0<0$. It will be completely reflected if $E<V_0$, and it will be completely transmitted only in the case $V_0=0$ (which is no step after all).

Part III: Calculation: [60 points] Choose three of the following four questions and perform the indicated calculations (20 points each).

14. A particle has wave function given by $\psi(x) = \begin{cases} 2\alpha^{3/2} x e^{-\alpha x} & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$ **This wave function is normalized. It has expectation value** $\langle p^2 \rangle = \hbar^2 \alpha^2$ **. A possibly useful integral appears below.**

(a) Find the expectation values $\langle x \rangle$, $\langle x^2 \rangle$, **and** $\langle p \rangle$ **for this wave function.**

Because the wave function is real, $\langle p \rangle = 0$, but we'll calculate it anyway, just for fun. For the others, we have

$$\langle x \rangle = \int \psi^*(x) x \psi(x)\,dx = 4\alpha^3 \int_0^\infty x x^2 e^{-2\alpha x} dx = \frac{4\alpha^2 3!}{(2\alpha)^3} = \frac{24\alpha^3}{16\alpha^4} = \frac{3}{2\alpha},$$

$$\langle x^2 \rangle = \int \psi^*(x) x^2 \psi(x)\,dx = 4\alpha^3 \int_0^\infty x^2 x^2 e^{-2\alpha x} dx = \frac{4\alpha^2 4!}{(2\alpha)^5} = \frac{96\alpha^3}{32\alpha^5} = \frac{3}{\alpha^2},$$

$$\langle p \rangle = \int \psi^*(x) \frac{\hbar}{i}\frac{d}{dx}\psi(x)\,dx = \frac{4\alpha^3\hbar}{i}\int_0^\infty x e^{-\alpha x}\frac{d}{dx}\left(x e^{-\alpha x}\right)dx = \frac{4\alpha^3\hbar}{i}\int_0^\infty x e^{-\alpha x}(1-\alpha x)e^{-\alpha x}dx$$

$$= \frac{4\alpha^3\hbar}{i}\int_0^\infty \left(x - \alpha x^2\right)e^{-2\alpha x}dx = \frac{4\alpha^3\hbar}{i}\left[\frac{1!}{(2\alpha)^2} - \frac{2!}{(2\alpha)^3}\right] = \frac{\alpha\hbar}{i}(1-1) = 0.$$

(b) What are the uncertainties Δx **and** Δp **for this wave function?**

These are given by

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{3}{\alpha^2} - \left(\frac{3}{2\alpha}\right)^2} = \frac{1}{\alpha}\sqrt{3 - \frac{9}{4}} = \frac{1}{\alpha}\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2\alpha},$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\hbar^2\alpha^2 - 0^2} = \hbar\alpha.$$

(c) Check that it satisfies the uncertainty relation.

We have $(\Delta x)(\Delta p) = \left(\frac{\sqrt{3}}{2\alpha}\right)(\hbar\alpha) = \frac{\sqrt{3}}{2}\hbar > \frac{1}{2}\hbar$, so it clearly satisfies it.

Possibly useful integral: $\int_0^\infty x^n e^{-Ax} dx = \frac{n!}{A^{n+1}}$, where $n! = n(n-1)(n-2)\cdots 2\cdot 1.$

15. An electron with mass 9.109×10^{-31} kg is trapped in a 1D infinite square well of length *L*. It is in the state *n* = 5 and is found to have an energy of 1.50×10^{-19} J.

(a) What is the length *L*?

We start with the formula for the energy and solve it for the length *L*:

$$E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2},$$

$$L^2 = \frac{\pi^2\hbar^2 n^2}{2mE_n} = \frac{\pi^2\left(1.055\times10^{-34}\ \text{J}\cdot\text{s}\right)^2 5^2}{2\left(9.109\times10^{-31}\ \text{kg}\right)\left(1.50\times10^{-19}\,\text{J}\right)} = 1.004\times10^{-17}\ \text{m}^2,$$

$$L = \sqrt{1.004\times10^{-17}\ \text{m}^2} = 3.17\times10^{-9}\ \text{m} = 3.17\ \text{nm}.$$

(b) What is the energy of the state *n* = 3 for this potential?

We just go back to the regular formula

$$E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2} = \frac{\pi^2\left(1.055\times10^{-34}\ \text{J}\cdot\text{s}\right)^2 3^2}{2\left(9.109\times10^{-31}\ \text{kg}\right)\left(1.004\times10^{-17}\ \text{m}^2\right)} = 5.400\times10^{-20}\ \text{J}.$$

With less work, we could have noted that $E_3/E_5 = 3^2/5^2$, so $E_3 = \frac{9}{25}E_5$.

(c) If the electron starting in *n* = 5 absorbed 1.44×10^{-19} J, what would be the final *n* value?

We note that it started with energy 1.50×10^{-19} J, so adding this to the energy added we would have 2.94×10^{-19} J. Rearranging the same equation and solving for *n*, we have

$$n^2 = \frac{2mL^2E_n}{\pi^2\hbar^2} = \frac{2\left(9.109\times10^{-31}\ \text{kg}\right)\left(1.004\times10^{-17}\ \text{m}^2\right)\left(2.94\times10^{-19}\ \text{J}\right)}{\pi^2\left(1.055\times10^{-34}\ \text{J}\cdot\text{s}\right)^2} = 48.99,$$

$$n = \sqrt{48.99} = 7.00.$$

Because *n* must be an integer, the final result is *n* = 7, and any discrepancy is due to rounding. We could have gotten the final answer more quickly by noting that $E_n/E_5 = n^2/5^2 = 2.94/1.50$, so then $n^2 = 25\left(2.94/1.50\right) = 25\left(1.96\right) = 49$.

16. Electrons of mass $m = 9.109\times10^{-31}\text{ kg} = 5.11\times10^{5}\text{ eV}/c^2$ have an energy of $E = 3.00\text{ eV}$ and impact a barrier of width $L = 0.270\text{ nm} = 2.70\times10^{-10}\text{m}$ and potential of height $V_0 = 10.00\text{ eV}$.

(a) What is the damping coefficient α in m^{-1} or nm^{-1}?

The damping coefficient is

$$\alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar} = \frac{\sqrt{2\left(5.11\times10^{5}\text{ eV}\right)\left(10.0\text{ eV} - 3.00\text{ eV}\right)}}{\hbar c}$$

$$= \frac{2675\text{ eV}}{\left(6.582\times10^{-16}\text{ eV}\cdot\text{s}\right)\left(2.998\times10^{8}\text{ m/s}\right)} = 1.356\times10^{10}\text{ m}^{-1} = 13.56\text{ nm}^{-1}.$$

(b) What is the probability that the electrons manage to penetrate the barrier?

The transmission probability is given by

$$T = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\alpha L} = 16\frac{3.00\text{ eV}}{10.00\text{ eV}}\left(1 - \frac{3.00\text{ eV}}{10.00\text{ eV}}\right)\exp\left[-2\left(13.56\text{ nm}^{-1}\right)\left(0.270\text{ nm}\right)\right]$$

$$= 3.36e^{-7.32} = 2.22\times10^{-3} = 0.222\%\ .$$

(c) The barrier width is now increased, and it is now found that only 4.27×10^{-5} of the electrons make it through. What is the new width *L*?

The formula is the same, and the value of α is the same, so we simply need to solve for L:

$$4.27\times10^{-5} = T = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\alpha L} = 16\frac{3.00\text{ eV}}{10.00\text{ eV}}\left(1 - \frac{3.00\text{ eV}}{10.00\text{ eV}}\right)e^{-2\alpha L} = 3.36e^{-2\alpha L},$$

$$e^{-2\alpha L} = \frac{4.27\times10^{-5}}{3.36} = 1.27\times10^{-5},$$

$$-2\alpha L = \ln\left(1.27\times10^{-5}\right) = -11.27,$$

$$L = \frac{11.27}{2\alpha} = \frac{11.27}{2\left(13.56\text{ nm}^{-1}\right)} = 0.416\text{ nm} = 4.16\times10^{-10}\text{ m}.$$

17. An electron in hydrogen is measured and found to have total orbital angular momentum $L^2 = 12\hbar^2$.

(a) What is the value of the angular momentum quantum number *l*? What values for the *z*-component L_z are possible?

The total angular momentum satisfies $L^2 = \hbar^2\left(l^2 + l\right)$, and therefore $l^2 + l = 12$. This equation is solved by inspection to yield $l = 3$. Though $l = -4$ also works, only positive integers are allowed.

For the *z*-component of angular momentum we have $L_z = \hbar m$, where *m* takes on values from $-l$ to $+l$. Hence the possible values are $-3\hbar, -2\hbar, -\hbar, 0, \hbar, 2\hbar, 3\hbar$.

(b) Suppose the *total* angular momentum around the *z*-axis were measured, defined as $J_z = L_z + S_z$. What is the largest (most positive) and smallest (most negative) values that J_z could be?

The spin S_z only takes values $\pm\frac{1}{2}\hbar$. To get it most positive, we would choose $L_z = +3\hbar$ and $S_z = +\frac{1}{2}\hbar$ so that $J_z = \frac{7}{2}\hbar$, and to get it most negative, we would choose $L_z = -3\hbar$ and $S_z = -\frac{1}{2}\hbar$ so that $J_z = -\frac{7}{2}\hbar$.

(c) Suppose that J_z has the value $J_z = +\frac{3}{2}\hbar$. Argue that there are two possible values of the pairs L_z and S_z that allow this to happen.

The value of S_z is $\pm\frac{1}{2}\hbar$, so that $L_z = J_z - S_z = +\frac{3}{2}\hbar \mp \frac{1}{2}\hbar$. The two ways to achieve it are therefore $\left(L_z, S_z\right) = \left(\hbar, \frac{1}{2}\hbar\right)$ and $\left(L_z, S_z\right) = \left(2\hbar, -\frac{1}{2}\hbar\right)$.

(d) Given the *l*-value you found in part (a), what is the lowest (most negative) energy value E_n that the atom could have?

We recall that we must have $n > l = 3$. The lowest energy comes from the smallest *n*, or *n* = 4. The energy is then

$$E_4 = -\frac{\left(13.6\ \text{eV}\right)Z^2}{n^2} = -\frac{13.6\ \text{eV}}{4^2} = -0.850\ \text{eV}.$$