Name ___________________

Solutions to Final Exam
December 10, 2018

This test consists of five parts. Please note that in parts II through V, you can skip one question.

Part I: Multiple Choice (mixed new and review questions) [50 points] (2 points each)
For each question, choose the best answer

1. When an atom makes a transition from some level to another level, what normally occurs?
   A) A photon is emitted, whether it goes up in energy or down in energy
   B) A photon is absorbed, whether it goes up in energy or down in energy
   C) A photon is absorbed if it goes up in energy, and emitted if it goes down in energy
   D) A photon is emitted if it goes up in energy, and absorbed if it does down in energy
   E) None of the above

2. The energy of a photon with wavelength λ and frequency f is given by
   A) hf B) $h\lambda$ C) h/f D) f/h E) $f\lambda$

3. Below are the masses of the isotopes of Tungsten in u. Which one is ^{183}W?
   A) 179.9467 B) 181.9482 **C) 182.9502** D) 183.9509 E) 185.9544

4. For a perfect black body distribution, what determines the wavelength with the most power in it?
   A) It is directly proportional to the temperature
   B) It is inversely proportional to the temperature
   C) It is directly proportional to the energy density
   D) It is directly proportional to the frequency with the most power
   E) None of the above

5. If you calculate the expectation value of the position, $\langle x \rangle$, what does the result tell you?
   A) Where the particle will definitely be found
   B) The average location for the particle if you repeat the measurement many times
   C) The most likely place to find the particle
   D) The least likely place to find the particle
   E) The range of x-values where one might find the particle

6. When we say that an electron is in a 4d subshell of hydrogen, what do the 4 and d tell us?
   A) Energy level and total angular momentum
   B) Energy level and angular momentum around z-axis
   C) Energy level and spin
   D) Total angular momentum and angular momentum around z-axis
   E) Total angular momentum and spin

7. An object made of three anti-quarks would be considered a(n)
   A) Meson B) Baryon C) Lepton D) Neutrino **E) Anti-baryon**

8. Which of the following is always conserved in special relativity?
   A) Energy (only)
   B) Momentum (only)
   C) Mass (only)
   D) Energy and momentum, but not mass
   E) Energy, momentum, and mass

9. Which of the following might be the diameter of the nucleus of an atom?
   A) 3×10^{-9} m B) 3×10^{-11} m C) 3×10^{-13} m **D)** 3×10^{-15} m E) 3×10^{-17} m

10. For a very heavy stable (or nearly) stable nucleus, what fraction of the nucleons would one expect to be protons?
    A) 30% **B) 40%** C) 50% D) 60% E) 70%

11. White dwarfs are held up by what force?
    A) Ideal gas pressure
    B) Radiation pressure
    C) Electron degeneracy pressure
    D) Neutron degeneracy pressure
    E) Microscopic but very strong invisible gnomes

12. If (Z,A) is the charge and mass of a nucleus, what does it change to if we undergo β^+ decay?
    A) $(Z,A+1)$ B) $(Z,A-1)$ **C)** $(Z-1,A)$ D) $(Z+1,A)$ E) (Z,A) unchanged

13. According to one observer, a clock moving at high velocity compared to her will run
    A) Slower
    B) Faster
    C) The same speed
    D) Slower if it is the observer that is moving; faster if it is the clock that is moving
    E) Faster if it is the observer that is moving; slower if it is the clock that is moving

14. When electrons shift levels in ^{2}H, they have slightly different energies than in ^{1}H. Why?
    A) The electrons in ^{2}H have a slightly different mass
    B) The extra neutron in ^{2}H has a little bit of charge (normally ignored) that causes this shift
    C) Though the extra neutron is neutral, it does have magnetic interactions with the electron, which cause this shift
    D) The nucleus is a slightly different size, which changes the interaction with the electron
    E) As the electron orbits, the nucleus moves a little too, so you have to use the reduced mass, which depends (slightly) on the mass of the nucleus

15. In general relativity, what causes curvature?
    A) Energy density (only)
    B) Pressure (only)
    C) Mass (only)
    D) The stress-energy tensor, which include things like energy density and pressure
    E) Light

16. The Sun is what kind of star?
    A) Pre-main sequence **B) Main sequence**C) Red giant D) White dwarf E) Neutron star

17. Light coming from the surface of a high gravity object like a neutron star will be changed in what way, according to general relativity?
    A) It will be shifted to a shorter wavelength
    B) It will be shifted to a longer wavelength
    C) It will remain as light with no change in wavelength
    D) It will be converted to neutrinos
    E) It will fail to escape, because it can't cross the event horizon

18. The reason that the strong force that holds the nucleus together is short range is because
    A) The particles that carry it (pions) are massive rather than massless
    B) Nucleons interact via direct contact, not by exchanging particles
    C) A cloud of screening particles blocks out interactions at long distances
    D) The total strong charge of a nucleus is zero, so there's no attraction far away
    E) None of these is correct

19. In special relativity, there is a fourth component of momentum, which is
    A) Time B) Mass C) Distance D) Angular Momentum **E) Energy**

20. Which rule is true about fermions in particle physics interactions?
    A) The total number of fermions is always unchanged
    B) The number of fermions on each side of the interaction must be odd
    C) The number of fermions on each side of the interaction must be even
    D) The total number of fermions (left plus right) must be odd
    E) The total number of fermions (left plus right) must be even

21. What do all main sequence stars have in common?
    A) They all have about the same surface temperature
    B) They all have about the same mass
    C) They all have about the same radius
    D) They all have about the same luminosity
    E) They are all converting hydrogen to helium at their centers

22. For the infinite square well with allowed region $0 < x < a$, which of the following formulas makes sense, given that the boundary is at 0?
    A) $\psi = e^{ax}$ B) $\psi = e^{-ax}$ C) $\psi = \cos(kx)$ **D)** $\psi = \sin(kx)$ E) $\psi = e^{-Ax^2}$

23. Which of the following formulas is still true in special relativity?
   A) $F = ma$ B) $E = \frac{1}{2}mv^2$ **C)** $W = Fd$ D) $p = mv$ E) None of these is correct

24. Which sorts of objects have produced gravity waves that have been measured?
   A) Rotating but otherwise stationary black holes
   B) Merging black holes
   C) Exploding stars
   D) Particles colliding at high speeds in particle colliders
   E) Gravity waves have not yet been detected

25. Which of the following particles has strong interactions?
   A) Electron B) Photon C) Neutrino **D) Proton** E) None of these

Part II: Short answer (review material) [20 points] (10 points each)
Choose **two** of the following three questions and give a short answer (1-3 sentences)

26. According to special relativity, how do moving objects compare in length to objects at rest? Can this be used to determine which of two observers is actually moving, and which is at rest?

As viewed by a stationary observer, a moving object looks shorter (but only in the direction of motion). However, everything is relative, so a moving observer would see a stationary object as shorter. You can't determine which one is actually moving, because each view is equally valid in special relativity.

27. What assumptions in the Bohr model of the atom allow one to explain why different atoms absorb or emit light at only specific wavelengths. Equations are not necessary.

The Bohr model assumes that electrons in atoms can only be at certain specific energy levels, depending on the type of atom it is. It also assumes that when an atom moves up or down in energy level, the atom absorbs or emits a single photon carrying all of the energy difference. This explains why each type of atom has its own specific set of frequencies (and hence wavelengths) associated with it.

28. Explain qualitatively what happens, according to quantum mechanics, when a particle of energy *E* impacts a barrier of finite width with a potential energy $V_0 > E$.

As the wave function impacts the barrier, it begins to exponentially damp away. However, if the barrier has a finite width, there is a small wave on the far side of the barrier, indicating a small but non-zero chance that it makes it through the barrier.

Part III: Short answer (new material) [30 points] (10 points each)

Choose **three** of the following four questions and give a short answer (1-3 sentences) .

29. Explain what a half-life is. In particular, if we started with 100,000 atoms of some isotope, how many would be left after <u>two</u> half-lives. Explain how you can find out the decay rate λ given the half-life.

Half-life is the amount of time that it takes half of a substance to disappear. In two half-lives, the amount will be divided by two twice, or divided by four, so there will be 25,000 atoms of that remaining. The decay rate is related to the half-life by the formula $\lambda t_{1/2} = \ln 2$.

30. Name at least five quarks in the standard model. Name at least four of the force carrying particles in the standard model. Finally, name the particle in the standard model which is responsible for all masses.

The six quarks are the up, down, strange, charm, top and bottom quarks. The five force carrying particles are the photon (γ), the gluon, the W^+ and W^- and the Z (the graviton carries gravity, but is not in the standard model). The particle responsible for mass is the Higgs boson.

31. The main sequence of stars forms a diagonal band on the Hertzsprung-Russel diagram. How do the stars in the upper left compare with those in the lower right in terms of temperature, luminosity, and mass compare with the lower right. Is the Sun on the main sequence, and where?

Stars in the upper left of the Hertzsprung-Russel diagram are somewhat hotter, much more luminous, and mor massive than stars in the lower right of the diagram. The Sun is on the main sequence, towards the middle.

32. "Matter tells space how to curve, and matter tells space how to move." Explain this statement. Your answer should probably include words like "stress-energy tensor" and "geodesic."

Matter tells space how to curve via the stress-energy tensor, which is a measure of the energy density, momentum density, and other quantities like the stress. This is equated to the Einstein tensor, which is a measure of curvature of the space-time.

Space tells matter how to move, because matter (in the absence of forces) follows geodesics, the path of longest proper time between any two points in spacetime.

Part IV: Calculation (review material) [40 points] (20 points each)

Choose **two** of the following three questions and perform the indicated calculations

33. Protons have a mass of 938 MeV/c^2. A particular proton, however, has an energy of $E = 1437\,\text{MeV}$.

(a) What is the velocity, both as a fraction of *c*, and in m/s?

The energy in general is given by $E = \gamma mc^2$, so we find

$$\gamma = \frac{E}{mc^2} = \frac{1437\text{ MeV}}{938\text{ MeV}} = 1.532 = \frac{1}{\sqrt{1 - v^2/c^2}},$$

$$1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2} = \frac{1}{1.532^2} = 0.426,$$

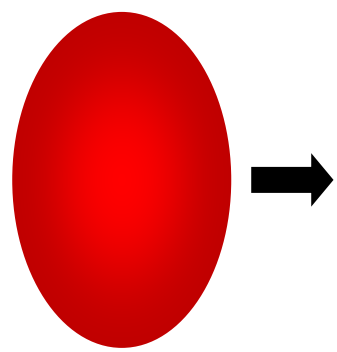
$$\frac{v^2}{c^2} = 1 - 0.4261 = 0.5739,$$

$$v = \sqrt{0.5739}c = 0.7576c = 0.7576\left(2.998\times10^8\text{ m/s}\right) = 2.271\times10^8\text{ m/s}.$$

(b) Assume a proton at rest is normally a sphere of diameter exactly 1.75 fm. If the proton is moving to the right, what shape will it have, and what will its diameter be in each of the three directions?

The proton becomes compressed in only in the direction of motion, so the diameter in the two transverse directions will remain the same. In the direction of motion, it will be Lorentz contracted by a factor of γ, so $L = L_p/\gamma$, so it will have diameter (in this direction)

$$d = \frac{d_p}{\gamma} = \frac{1.75\text{ fm}}{1.532} = 1.14\text{ fm}.$$



The shape is sketched at right, with one inch representing 1 fm.

(c) The proton goes at this speed for $4.80\text{ ns} = 4.80\times10^{-9}\text{ s}$, as measured in its own frame. How far does it go, as measured by us?

Because time slows down, this period of time will be longer as viewed by us, according to the formula $\Delta t = \gamma\tau$, so as viewed by us the time it takes is

$$\Delta t = \gamma\tau = 1.532\times4.80\text{ ns} = 7.35\text{ ns}.$$

We then simply mutliply by the velocity to get the distance,

$$d = v\Delta t = \left(2.271\times10^8\text{ m/s}\right)\left(7.35\times10^{-9}\text{ s}\right) = 1.67\text{ m}.$$

34. A particle has wave function $\psi(x) = \begin{cases} N\sin(3\pi x/a) & 0 \le x \le a, \\ 0 & \text{otherwise,} \end{cases}$

where *N* is an unknown normalization constant. Some possibly useful integrals are below.

(a) What is the value of the constant *N*?

The probability density integrated over the region must be 1, so we have

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = N^2 \int_0^a \sin^2(3\pi x/a)\, dx = \tfrac{1}{2} N^2 a,$$

$$N^2 = \frac{2}{a}, \quad \text{so} \quad N = \sqrt{\frac{2}{a}}.$$

(b) At what place(s) in the allowed region will the particle definitely not be?

The particle will not be at any place where the wave function vanishes, which is when $\sin(3\pi x/a) = 0$. Sine vanishes at multiples of π, so we have $3\pi x/a = n\pi$, $x = \tfrac{1}{3}an$. Of course, we are only interested in the places in the allowed region, which correspond to n from 0 to 3, so the points are $x \in \left\{0, \tfrac{1}{3}a, \tfrac{2}{3}a, a\right\}$. The first and last are just the endpoints of course.

(c) What is the most probable place(s) to find the particle?

The particle is most likely to be at the points where the wave function is at a (positive) maximum or (negative) minimum, which is when its derivative vanishes. We therefore have

$$0 = \frac{d}{dx} N \sin\left(\frac{3\pi x}{a}\right) = \frac{3\pi N}{a} \cos\left(\frac{3\pi x}{a}\right),$$

Cosine vanishes at odd mutiples of $\tfrac{1}{2}\pi$, so $3\pi x/a = \tfrac{1}{2}\pi(2n+1)$ and hence $x = \tfrac{1}{6}a(2n+1)$. The poins in the allowed region correspond to n running from 0 to 2, so $x \in \left\{\tfrac{1}{6}a, \tfrac{1}{2}a, \tfrac{5}{6}a\right\}$.

(d) What is the probability density that the particle is at the point $x = \tfrac{1}{9}a$**?**

The probability density is just the square of the magnitude of the wave functions, so

$$\rho\left(\tfrac{1}{9}a\right) = \left|\psi\left(\tfrac{1}{9}a\right)\right|^2 = N^2 \sin^2\left(\frac{3\pi \tfrac{1}{9}a}{a}\right) = \frac{2}{a}\sin^2\left(\frac{\pi}{3}\right) = \frac{2}{a}\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{2a}.$$

Possibly useful integrals: $\int_0^a \sin(n\pi x/a)\, dx = \dfrac{a}{\pi n}\left[1 - (-1)^n\right]$, $\int_0^a \sin^2(n\pi x/a)\, dx = \dfrac{1}{2}a$

35. A helium atom ($Z = 2$) with one electron in it has an energy of $E = -3.40\text{ eV}$.

(a) What is the principle quantum number n for this electron?

Since there is only one electron, it is a hydrogen-like atom, which means that we can use the formula

$$-3.40\text{ eV} = E_n = -\frac{(13.6\text{ eV})Z^2}{n^2} = \frac{-(13.6\text{ eV})2^2}{n^2},$$

$$n^2 = \frac{(13.6\text{ eV})2^2}{3.40\text{ eV}} = 16.00,$$

$$n = 4.$$

The principal quantum number n must be an integer, so the result must be exactly four.

(b) What are the possible values that the orbital angular momentum squared L^2, the z-component L_z, the total spin square S^2, and the z-component of the spin S_z take on, based on the value of n you found in part (a)? In each case, your answer might be a single value or a list of values.

The angular momentum quantum number l must be smaller than n, so it can only take on the values $l = 0, 1, 2,$ or 3. The resulting values for L^2 must then be

$$L^2 = \hbar^2\left(l^2 + l\right) = 0, 2\hbar^2, 6\hbar^2, \text{or } 12\hbar^2.$$

For L_z, m can take on values ranging from $-l$ to $+l$, but since l can be as large as 3, this means m can run from -3 to +3. Consequently the values L_z can take on are

$$L_z = \hbar m = -3\hbar, -2\hbar, -\hbar, 0, \hbar, 2\hbar, 3\hbar.$$

The total spin for an electron is $s = ½$, so $S^2 = \hbar^2\left(s^2 + s\right) = \frac{3}{4}\hbar^2$. Finally, the z-component of angular momentum is always

$$S_z = m_s\hbar = \pm\tfrac{1}{2}\hbar.$$

(c) The angular momentum around the z axis is observed to be $L_z = -2\hbar$. Based on this additional piece of information, what would you change about your answers to part (b), if anything?

Obviously, the value of L_z is now $L_z = -2\hbar$. But also, since this implies $m = -2$, it follows that since m runs from $-l$ to $+l$, l must now be at least as big as m, so $l = 2$ or more, so 2 or 3. Therefore the only possible values for L^2 would now be

$$L^2 = \hbar^2\left(l^2 + l\right) = 6\hbar^2 \text{ or } 12\hbar^2.$$

The other quantities would be unaffected.

Part V: Calculation (new material): [60 points]
Choose **three** of the following four questions and perform the calculations (20 points each)

36. The isotope ^{238}U decays via α decay. This is followed by β^- decay, β^- decay, and α decay. For this problem, you may fill in your answers directly on the table, if you want. A table of isotopes appears on the last page of this test.

Nucleus	*A*	*Z*	*N*	Decay	*Q* (MeV)
^{238}U	238	92	146	α	4.28
^{234}Th	234	90	144	β^-	0.28
^{234}Pa	234	91	143	β^-	2.24
^{234}U	234	92	142	α	4.84
^{230}Th	230	90	Most Unstable=^{234}Pa		

(a) Write the values of *Z* and *A* for the initial nucleus, and for the resulting nucleus after each of the four steps. Note that in each step, the daughter nucleus becomes the parent for the next step. Then fill in the type of nucleus (^{238}U, etc.)

For α decay, *Z* decreases by 2 and *A* by 4. For β^- decay, it leaves *A* unchanged while *Z* increases by 1. Since we start at $(Z,A)=(92,238)$, this means the chain of *Z* and *A* values is

$$(Z,A)=(92,238)\to(90,234)\to(91,234)\to(92,234)\to(90,230)$$

Since element 90 is Th and element 91 is Pa, we have $^{238}\text{U}\to\ ^{234}\text{Th}\to\ ^{234}\text{Pa}\to\ ^{234}\text{U}\to\ ^{230}\text{Th}$.

(b) For each <u>parent</u> nucleus, calculate the number of neutrons (*N*). Based on what you know about stability of nuclei vs. even or odd, which of these nuclei do you think is least stable and decays most rapidly?

The value of *Z* has already been given, and the value of *N* is just $N=A-Z$. We have included the numbers in the table. Because nuclei with an odd number of protons or neutrons are unstable, we expect the ^{234}Pa to be the most unstable.

(c) What is the *Q*-value for each step?

The *Q*-value for α decay is given by $Q=\left(M_P-M_D-M_{He}\right)c^2$, so we have

$$Q\left(^{238}\text{U}\to\,^{234}\text{Th}\right)=\left(238.0508-234.0436-4.0026\right)\text{u}c^2=\left(0.0046\right)\left(931.5\text{ MeV}\right)=4.28\text{ MeV},$$
$$Q\left(^{234}\text{U}\to\,^{230}\text{Th}\right)=\left(234.0409-230.0331-4.0026\right)\text{u}c^2=\left(0.0052\right)\left(931.5\text{ MeV}\right)=4.84\text{ MeV}.$$

For β^- decay, the formula is $Q=\left(M_P-M_D\right)c^2$, so we have

$$Q\left(^{234}\text{Th}\to\,^{234}\text{Pa}\right)=\left(234.0436-234.0433\right)\text{u}c^2=\left(0.0003\right)\left(931.5\text{ MeV}\right)=0.28\text{ MeV},$$
$$Q\left(^{234}\text{Pa}\to\,^{234}\text{U}\right)=\left(234.0433-234.0409\right)\text{u}c^2=\left(0.0024\right)\left(931.5\text{ MeV}\right)=2.24\text{ MeV}.$$

37. The Ω^- is a baryon with mass 1672 MeV/c^2 and strangeness –3. Its most common decay is $\Omega^- \to \Lambda^0 K^?$, where Λ^0 and $K^?$ each have strangeness –1 , the Λ^0 has a mass of 1116 MeV/c^2 , and the $K^?$ is a meson.

(a) Is the Λ^0 a baryon, an anti-baryon, or a meson?

Since there is a baryon on the left, there must be a baryon on the right, for baryon number to be conserved. Since the *K* is a meson, the other particle, the Λ^0 is a baryon.

(b) What is the charge of the $K^?$? What is its maximum possible mass?

The Ω^- from its name has charge –1, so that must be the total charge on the right. Since the Λ^0 is neutral, the $K^?$ must have charge –1. Its actual name is K^-, not $K^?$.

Since this process is a decay, the mass of the initial particle must exceed the sum of the masses on the right, so $m_\Omega > m_\Lambda + m_K$, so that

$$m_K < m_\Omega - m_\Lambda = \left(1672 \text{ MeV/}c^2\right) - \left(1116 \text{ MeV/}c^2\right) = 556 \text{ MeV/}c^2 .$$

(c) Is this interaction, strong, weak, or electromagnetic?

Since this decay actually occurs, we can skip all the checks to see if it is possible. The next step is to look at conservation of strangeness. The strangeness on the left is -3, while the strangeness on the right is $-1-1=-2$, so strangeness is not conserved. This means this is a weak interaction.

(d) Only up, down, and strange quarks (or their anti-quarks) are found in these three particles. What is the quark composition of each of these particles?

The up quark has charge $+\frac{2}{3}$, while the down and strange quarks have charge $-\frac{1}{3}$. The strangeness tells you the number of strange quarks, since each strange quark has strangeness -1. The Ω^-, with strangeness -3 has three strange quarks, and as a baryon must have three quarks, so it is $\Omega^- = [sss]$. The Λ^0 with strangeness -1, so it has one strange quark, and as a baryon so it must have three quarks. The only way to make the charge come out to zero is if $\Lambda^0 = [uds]$, so that the total charge is $+\frac{2}{3}-\frac{1}{3}-\frac{1}{3}=0$. Finally, the K^- is a meson with strageness -1 and charge -1. It must have one strange quark, plus and anti-quark, and the combination $K^- = [s\bar{u}]$ would then have the correct charge, with $-\frac{1}{3}-\left(+\frac{2}{3}\right)=-1$.

38. The star Betelgeuse has a luminosity of approximately $L = 120{,}000 L_\odot$ ($L_\odot = 3.828\times10^{26}$ W) and a surface temperature of $T = 3590$ K ($T_\odot = 5772$ K).

(a) What is the radius of Betelgeuse, both compared to the Sun's radius $R_\odot = 6.957\times10^5$ km, and as compared to the Sun-Earth distance ($\mathrm{AU} = 1.496\times10^8$ km)?

We use the formula relating radius, temperature and luminosity, $L/L_\odot = (R/R_\odot)^2 (T/T_\odot)^4$, and solve for *R*:

$$\left(\frac{R}{R_\odot}\right)^2 = \left(\frac{L}{L_\odot}\right)\left(\frac{T}{T_\odot}\right)^{-4} = (120{,}000)\left(\frac{3590}{5772}\right)^{-4} = 801{,}900\,,$$

$$R = \sqrt{801{,}900}\,R_\odot = 895 R_\odot\,,$$

$$R = 895\left(6.957\times10^5\ \mathrm{km}\right)\frac{\mathrm{AU}}{1.496\times10^8\ \mathrm{km}} = 4.16\ \mathrm{AU}\,.$$

(b) What is the radiation pressure near the surface, assuming it is a black body, in N/m^2?

We simply use the formula for radiation pressure, so we have

$$P_r = \frac{\pi^2 (k_B T)^4}{45(\hbar c)^3} = \frac{\pi^2\left[\left(1.381\times10^{-23}\ \mathrm{J/K}\right)\left(3590\,\mathrm{K}\right)\right]^4}{45\left[\left(1.055\times10^{-34}\ \mathrm{J\cdot s}\right)\left(2.998\times10^8\ \mathrm{m/s}\right)\right]^3} = 0.0419\ \mathrm{J/m^3} = 0.0419\ \mathrm{N/m^2}\,.$$

(c) Betelgeuse is about 222 pc away ($1\,\mathrm{pc} = 3.086\times10^{16}$ m). Assuming Betelgeuse radiates power equally in all directions, what is the flux (W/m^2) of radiation here?

If power is radiated equally in all directions, we would simply divide the total lumonisity by the area of a sphere of size *d*, which is given by $A = 4\pi d^2$. Hence the flux is

$$\mathcal{F} = \frac{L}{4\pi d^2} = \frac{1.2\times10^5\times3.828\times10^{26}\ \mathrm{W}}{4\pi\left(222\times3.086\times10^{16}\ \mathrm{m}\right)^2} = 7.79\times10^{-8}\ \mathrm{W/m^2} = 77.9\ \mathrm{nW/m^2}\,.$$

39. A researcher is studying the black hole at the center of a galaxy with unknown mass. He spends two days, as measured by him, at a distance of 982 AU ($\text{AU} = 1.496\times10^{11}\ \text{m}$ **) from the center of the black hole.**

(a) When he rejoins his companions far from the black hole, he discovers that his clock is off from other clocks by exactly one hour. Will it be "slow" (i.e., behind them by an hour) or "fast" (i.e., ahead of them by an hour).

Time runs slower near a black hole, and hence his clock will be slow.

(b) What is the Schwarzschild radius for this black hole, in AU?

The relationship between the proper time and the time as measured outside is $\tau = t\sqrt{1-2GM/c^2r}$. Using the formula $R_s = 2GM/c^2$, we see that this can be simplified to $\tau = t\sqrt{1-R_s/r}$. We now solve this for R_s:

$$\sqrt{1-\frac{R_s}{r}} = \frac{\tau}{t},$$

$$1-\frac{R_s}{r} = \frac{\tau^2}{t^2},$$

$$R_s = r\left(1-\tau^2/t^2\right).$$

Since he measured two days, we have $\tau = 48\ \text{h}$, and external clocks measure one hour more, or $t = 49\ \text{h}$. Putting this together, we have

$$R_s = (982\ \text{AU})\left(1-48^2/49^2\right) = 39.7\ \text{AU} = (39.7\ \text{AU})\left(1.496\times10^{11}\ \text{m/AU}\right) = 5.94\times10^{12}\ \text{m}.$$

(c) What is the mass of the black hole, in solar masses ($M_\odot = 1.989\times10^{30}\ \text{kg}$ **)?**

We now just solve the equation for the Schwarzschild radius, $R_s = 2GM/c^2$, for the mass:

$$M = \frac{c^2R_s}{2G} = \frac{\left(2.998\times10^8\ \text{m/s}\right)^2\left(5.94\times10^{12}\ \text{m}\right)}{2\left(6.673\times10^{-11}\text{m}^3/\text{kg}/\text{s}^2\right)} = \frac{4.00\times10^{39}\ \text{kg}}{1.989\times10^{30}\ \text{kg}/M_\odot} = 2.01\times10^9\ M_\odot.$$

Equations

Constants:

$$h = 6.626\times10^{-34}\ \text{J}\cdot\text{s} = 4.136\times10^{-15}\ \text{eV}\cdot\text{s}$$
$$\hbar = 1.055\times10^{-34}\ \text{J}\cdot\text{s} = 6.582\times10^{-16}\ \text{eV}\cdot\text{s}$$
$$G = 6.673\times10^{-11}\,\text{m}^3/\text{kg}/\text{s}^2$$
$$N_A = 6.022\times10^{23}$$
$$k_B = 1.381\times10^{-23}\ \text{J/K} = 8.671\times10^{-5}\ \text{eV/K}$$

$$\text{u} = 931.5\ \text{MeV}/c^2$$
$$\text{u} = 1.661\times10^{-27}\,\text{kg}$$
$$2m_ec^2 = 1.022\ \text{MeV}$$
$$M_{\text{He}} = 4.0026\ \text{u}$$

Hydrogen-Like atoms: $E_n = -\dfrac{(13.6\ \text{eV})Z^2}{n^2}$

Radiation Pressure: $P_r = \dfrac{\pi^2(k_BT)^4}{45(\hbar c)^3}$ Stellar Luminosity: $\dfrac{L}{L_\odot} = \left(\dfrac{R}{R_\odot}\right)^2\left(\dfrac{T}{T_\odot}\right)^4$

Gravitational time dilation: $\tau = t\sqrt{1-\dfrac{2GM}{c^2r}}$ Schwarzschild radius: $R_S = \dfrac{2GM}{c^2}$

Isotope Masses

Z	Element	Sym	Mass#	Atomic Mass
90	Thorium	Th	227	227.0277
			228	228.0287
			229	229.0318
			230	230.0331
			231	231.0363
			232	232.0381
			234	234.0436
91	Protactinium	Pa	231	231.0359
			234	234.0433
92	Uranium	U	231	231.0363
			232	232.0371
			233	233.0396
			234	234.0409
			235	235.0439
			236	236.0456
			238	238.0508
			239	239.0543
93	Neptunium	Po	235	235.0441
			236	236.0466
			237	237.0482