Solutions to the Final
December 11, 2020

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered.

Part I: Multiple Choice [50 points]
For each question, choose the best answer (2 points each). Your test will have had these questions in a random order.

2. Which of the following is true in the proximity of a large mass, according to general relativity?
   A) The circumference of a circle of radius r is still $2\pi r$
   B) The gravitational acceleration of an object will depend on its mass
   C) Time slows down
   D) Time speeds up
   E) Spacetime is still perfectly flat

3. Under what circumstances is it wise to solve a problem using spherical coordinates (r,θ,ϕ) instead of Cartesian coordinates (x,y,z)
   A) When the potential is spherically symmetric, so it depends only on the radius r
   B) When you are looking for solutions with no angular momentum, $l=0$.
   C) When it is unclear which axes to pick for x, y, and z.
   D) When we need the wave function to fall off at infinity in all directions
   E) When solving problems with no potential, like for plane wave solutions

4. Which of the following methods would be a good way to measure a gravity wave?
   A) Measure how the gravitational acceleration of objects near the Earth changes as it passes
   B) Measure distances with high accuracy, such as by using a very long interferometer
   C) Measuring how the mass of an object changes as it passes
   D) Carefully monitoring changes in the orbit of the Earth
   E) Checking for small electromagnetic waves generated by the gravity wave

5. In the most stable nuclei, the number of protons tends to be ________ and the number of neutrons tends to be ______ .
   A) Odd, odd **B) Even, even** C) Even, odd D) Odd, even E) Prime, composite

6. The particles that are exchanged inside the nucleus that hold the protons and neutrons together are called
   A) pions B) photons C) gravitons D) Z-bosons E) Nucleons

7. Which of the following is one of the assumptions made by the Bohr model?
   A) The electron does not have a definite position, but rather is described by a wave function
   B) All particles must satisfy the uncertainty relationship
   C) The number of waves that fit around the circular orbit must be an integer
   D) When an electron goes from one level to another in an atom, a single photon is emitted or absorbed
   E) The positive charge of an atom is spread throughout it, while electrons are stuck in this background; the "plum pudding" model

8. Under what conditions will two observers inevitably agree which of two events A and B came first?
   A) Only if they are timelike separated
   B) Only if they are spacelike separated
   C) They will always agree
   D) They will never be guaranteed to agree
   E) Only if both observers were present at both events

9. According to special relativity, all vectors should have four components. What is the fourth component of the vector corresponding to momentum?
   A) Mass B) Velocity C) Time D) Distance **E) Energy**

10. A ^{7}Be atom will decay via electron capture, but an isolated nucleus of ^{7}Be is stable. How is this possible?
    A) The negatively charged electrons in the atom pull out the decay products
    B) The electrons pull on the nucleus, making it larger and less stable
    C) The electrons can annihilate with the neutrinos that are produced, creating more energy
    D) Electron capture requires electrons that can be captured
    E) An atom without electrons will have a higher priority finding electrons rather than decaying

11. Which of the following is not generally conserved in relativity?
    A) Energy (only)
    B) Momentum (only)
    C) Mass (only)
    D) None of the above are conserved
    E) All of the above are conserved

12. Which event caused the universe to transform from being opaque to transparent?
    A) Neutron/proton freezeout
    B) Electron/positron annihilation
    C) The Grand Unified Theory (GUT) transition
    D) Recombination
    E) Primordial Nucleosynthesis

13. The physical velocity some object moves at corresponds to, when you think of it as a wave, is
    A) The phase velocity
    B) The group velocity
    C) The speed of light
    D) The node velocity
    E) The anti-node velocity

14. The universe is currently flat, or very close to flat. Which early universe event is conjectured to be the reason for this?
    A) The growth of density perturbations
    B) Primordial nucleosynthesis
    C) The transition from radiation domination to matter domination
    D) Recombination
    E) Inflation, a period of rapid growth

15. What information did Rutherford glean by scattering α−particles off of gold and aluminum foil?
    A) The density of matter inside atoms is very low
    B) Most of the deflection was caused by the electrons
    C) Electrons are negatively charged
    D) The positive charge and most of the mass of an atom is spread throughout the atom
    E) The positive charge and most of the mass of an atom is concentrated in a tiny region of the atom

16. How many quantum states are there in a 3d subshell?
    A) 3 B) 4 C) 5 **D) 10** E) 12

17. Suppose I have two objects each of mass m, and they are moving towards each other at high velocity. The effective mass of the system of two masses will be
    A) Equal to $2m$
    B) Greater than $2m$
    C) Less than $2m$, but definitely positive
    D) Less than $2m$; it could even be negative or zero
    E) Insufficient information

18. The minimum positive energy that an electromagnetic wave of frequency f or angular frequency ω can have, according to Planck, is given by
    A) hf
    B) $\frac{1}{2}hf$
    C) $\frac{1}{2}\hbar\omega$
    D) $\frac{\pi^2\hbar^2}{2ma^2}$
    E) There is no minimum energy; all positive energies are possible

19. What is the difference between a protostar (pre-main sequence) and a main sequence star?
    A) Protostars are mostly helium; main sequence are mostly hydrogen
    B) Protostars are mostly hydrogen; main sequence are mostly helium
    C) The core of protostars are not hot enough for fusion; main sequence stars are hot enough
    D) Protostars mostly transport energy via radiation; main sequence mostly by convection
    E) Protostars have high mass; main sequence stars are low mass

20. Suppose you have a solution to Schrodinger's time-independent equation, but unfortunately the normalization integral $\int \left|\psi(x)\right|^2 dx = 4$. What can you do to fix the problem?
    A) Multiply the energy by four
    B) Divide the wave function by four
    C) Divide the wave function by two
    D) Multiply the wave function by two
    E) Nothing; if it isn't normalized properly you have to start over

21. The most important source of pressure for a white dwarf star is
    A) Ideal gas pressure
    B) Radiation pressure
    C) Electrostatic repulsion
    D) Degeneracy pressure from the nuclei
    E) Degeneracy pressure from the electrons

22. Iron has Z = 26, and a certain isotope of iron has a mass of 53.93961 u. How many neutrons does it have?
    A) 26 B) 27 **C) 28** D) 53 E) 54

23. According to Einstein's equations, the stress-energy tensor, which includes things like energy density and momentum density, is directly proportional to what?
    A) An object's acceleration
    B) An object's velocity
    C) An object's geodesic equation
    D) The Einstein tensor, a measure of curvature
    E) The rate of change of momentum of an object

24. If the wave function for a particle is given by $\psi(x)$, and the expectation value of an operator is given by $\langle O \rangle = \int \psi^*(x) O \psi(x) dx$, which of the following formulas would allow one to calculate the expectation value of the momentum?

A) $\int \frac{\hbar}{i} \frac{\partial}{\partial x} \psi(x) dx$

B) $\int \frac{\hbar}{i} \left[\frac{\partial}{\partial x} \psi^*(x) \right] \psi(x) dx$

C) $\int \frac{\hbar}{i} \frac{\partial}{\partial x} \left[\psi^*(x) \psi(x) \right] dx$

D) $\int \psi^*(x) \psi(x) \frac{\hbar}{i} \frac{\partial}{\partial x} dx$

E) $\int \psi^*(x) \frac{\hbar}{i} \frac{\partial}{\partial x} \psi(x) dx$

25. When we say the universe is homogeneous and isotropic, what do we mean?

A) It is the same at all places and in all directions

B) It is the same at all times

C) The same laws of physics apply everywhere, and all directions are equal

D) Most of the objects in the universe are approximately spherically symmetrical

E) The cosmic microwave background must have originated at all places and in all directions equally

26. The Hertzsprung-Russel diagram shows high temperatures where on the diagram?

A) Right

B) Left

C) Top

D) Bottom

E) The HR-diagram doesn't include temperature information

Part II: Short essay, review [20 points]

Choose two of the following three questions, and write a short essay (2-3 sentences). You may type both answers into the answer box at the end, or you may upload your answers as an image into the box. Each question is worth 10 points.

27A. Agree or disagree with this statement, and explain approximately what each observer would see: If two observers, *A* and *B*, are moving compared to each other, they can always figure out which one is actually moving by seeing which of their two clocks runs faster.

Disagree: According to the principles of relativity, there is no such thing as absolute motion, so it is impossible to use this to determine which is actually moving. Observer *A* will say that *B*'s clock is running more slowly, and observer *B* will say that *A*'s clock is running more slowly.

27B. Explain why it is that, when you shine a light on a piece of metal, electrons pop off (i) only if the wavelength is short enough, and (ii) immediately, even if the intensity of the light is low, rather than having to accumulate enough energy.

Light comes out in packets of energy *hf*. If the wavelength is long, the frequency will be low, and hence the energy will be low, too low to free any electrons. If the wavelength is short enough, then each photon has enough energy to free an electron, and since the energy is concentrated in packets, you don't have to build anything up, and it can immediately start freeing electrons.

27C. **Suppose that the angular momentum of an electron around the *z*-axis is given by $L_z = 4\hbar$. What is (i) the minimum possible value for the total angular momentum squared $\mathbf{L}^2$, and (ii) the possible values of the angular momentum S_z. (Feel free to write $\hbar$ as** hbar **as needed)**

Since $L_z = m\hbar$, we have $m = 3$. Since *l* is always at least as big as *m*, we must have *l* at least 3, and the minimum value of $\mathbf{L}^2$ will be $\mathbf{L}^2 = \hbar^2\left(l^2 + l\right) = \hbar^2\left(3^2 + 3\right) = 12\hbar^2$. The possible values of S_z are always $S_z = \pm\frac{1}{2}\hbar$.

Part III: Short essay, new material [30 points]

Choose three of the following four questions, and write a short essay (2-3 sentences). You may type all three answers into the answer box at the end, or you may upload your answers as an image into the box. Each question is worth 10 points.

28A. For the most stable isotopes, what fraction of the nucleons will be protons? Explain how this varies for light vs. heavy nuclei. What will happen to a nucleus that has too many protons or too few neutrons?

The most stable light nuclei ($A < 50$) are about 50% protons, while for heavy nuclei, ($A > 150$) they are closer to 40% protons, with intermediate values in between. If a nucleus has substantially more than this many protons, it can beta-plus decay or electron capture to decrease the protons, and if it has too few protons, it can beta-minus decay.

28B. Main sequence stars like the Sun burn what isotope to create what isotope? Give at least two reasons why this process is not very fast.

The process converts ^{1}H to ^{4}He in several complicated steps. It is a slow process because (i) the protons repel each other, (ii) they have to quantum tunnel to get through this repulsion, and (iii) they then have to undergo a weak decay.

28C. In the very early universe, there were approximately equal numbers of protons and neutrons. Explain qualitatively how this ratio changed, and what became of the vast majority of these protons and neutrons.

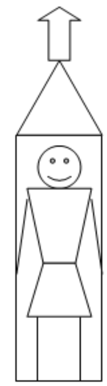
At high temperatures, protons and neutrons interconverted freely, but as the temperature dropped, the lower mass (and lower energy) protons were favored, and the balance shifted, so that there were about seven protons per neutrons. The neutrons ended up getting combined with the protons to make ^{4}He, while the remaining protons simply remained and became ^{1}H.

28D. According to Newton, Mercury's orbit around the Sun will be an ellipse. How, if it all, does this change according to Einstein's general theory of relativity?

Einstein’s general theory indicates that the shape will be slightly but not noticeably different than an ellipse, but more importantly, the direction of the major axis of the ellipse will *precess*, so that it gradually changes over time.

Part IV: Calculation, Old Material: [40 points]

Choose two of the following three questions. Each question is worth 20 points. Type **only your answers** to each part into the essay box provided.



29. A woman who has just become pregnant has a mass of 69.0 kg. She gets aboard a spacecraft traveling at a velocity of 2.83×10^8 m/s .

a) What is her total energy *E*, as measured by us?

We need to find the Lorentz factor, which is given by

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\left(\dfrac{2.83\times10^8\text{ m/s}}{2.998\times10^8\text{ m/s}}\right)^2}} = \frac{1}{\sqrt{1-0.944^2}} = 3.03.$$

The energy is just given by

$$E = \gamma mc^2 = 3.03(69.0\,\text{kg})(2.998\times10^8\text{ m/s})^2 = 1.879\times10^{19}\text{ J}.$$

b) Assuming her baby takes the usual 0.75 year to gestate, as measured by her, how long will it be until the baby is born, according to us?

The time measured by her is the proper time. We will see it as taking longer; according to us

$$\Delta t = \gamma\tau = 3.03\times(0.75\text{ y}) = 2.27\text{ y}.$$

c) How far will she have traveled in this amount of time? 1 y = 3.156×10^7 s

The distance she travels is simply velocity times time, so

$$d = v\Delta t = (2.27\text{ y})(3.156\times10^7\text{ s/y})(2.83\times10^8\text{ m/s}) = 2.03\times10^{16}\text{ m}.$$

d) The woman is 169 cm tall and measures 92 cm around her hips. What are the same two measurements, according to observers on the Earth? The rocket is traveling in the direction of her height.

The distance around her hips will be unchanged, so still 92 cm, because Lorentz contraction occurs only in the direction of motion. Her apparent height will be reduced to

$$L = \frac{L_p}{\gamma} = \frac{169\,\text{cm}}{3.03} = 55.8\text{ cm}.$$

30. The energy of Hydrogen-like atoms is given by $E_n = -\dfrac{13.6Z^2\ \text{eV}}{n^2}$**. A boron atom (*Z* =5) has a single electron in level *n* = 7.**

a) Suppose it were to fall to the *n* = 5 state. Would it emit or absorb a photon to do this? What would be the corresponding energy of that photon?

We start by substituting $Z = 5$, and we find $E_n = -(340\ \text{eV})/n^2$. Since it is dropping to a lower level, the atom is losing energy, which implies that it must **emit** energy in the form of a photon. The corresponding energy will be the difference of the initial and final energy, or

$$\Delta E = E_7 - E_5 = -\frac{340\ \text{eV}}{7^2} + \frac{340\ \text{eV}}{5^2} = -6.94\ \text{eV} + 13.6\ \text{eV} = 6.66\ \text{eV}.$$

This is well into the ultraviolet.

b) What would be the wavelength of the corresponding photon from part (a)?

We can get the frequency from $E = hf$ and then the wavelength from $\lambda f = c$, or, better yet, combine them in a single step:

$$\lambda = \frac{c}{f} = \frac{hc}{hf} = \frac{hf}{E} = \frac{1239.8\ \text{eV}\cdot\text{nm}}{6.66\ \text{eV}} = 186\ \text{nm}.$$

c) Suppose the atom, starting in level 7, absorbed a photon of wavelength 452 nm. What would be the resulting level *n* of the atom after absorption of the photon?

The first step is to get the energy of this photon, which we can use by rearranging the formula we found $\lambda = hc/E$, so we have

$$E = \frac{hc}{\lambda} = \frac{1239.8\ \text{eV}\cdot\text{nm}}{452\ \text{nm}} = 2.743\ \text{eV}.$$

We add this to the starting energy to get the final energy

$$E_n = E_7 + \Delta E = -\frac{340\ \text{eV}}{7^2} + 2.74\ \text{eV} = -6.94\ \text{eV} + 2.74\ \text{eV} = -4.20\ \text{eV}.$$

We then solve the equation for *n*:

$$-\frac{340\ \text{eV}}{n^2} = -4.20\ \text{eV}.$$

$$n^2 = \frac{340\ \text{eV}}{4.20\,\text{eV}} = 80.9,$$

$$n = \sqrt{80.9} = 9.00.$$

Since *n* must be an integer, and we got lucky with rounding, we simply truncate this to *n* = 9.

31. A particle in one dimension is in the region $0 \le x \le a$ and has wave function

$$\psi(x) = a^{-2}\sqrt{12x}\,(a-x) = a^{-2}\sqrt{12}\left(ax^{1/2} - x^{3/2}\right).$$

a) Where in the allowed region is it impossible for the particle to be?

It is impossible for the particle to be anywhere the wave function vanishes, so when $0 = a^{-2}\sqrt{12x}\,(a-x)$, which implies either $x = 0$ or $x = a$.

b) Where in the allowed region is the particle most likely to be?

It is most likely to be anywhere where the wave function has a maximum or minimum, which will be where the derivative vanishes, so when

$$0 = \frac{d}{dx}\psi(x) = a^{-2}\sqrt{12}\frac{d}{dx}\left(ax^{1/2} - x^{3/2}\right) = a^{-2}\sqrt{12}\left(\tfrac{1}{2}ax^{-1/2} - \tfrac{3}{2}x^{1/2}\right),$$

$$\tfrac{1}{2}ax^{-1/2} = \tfrac{3}{2}x^{1/2},$$

$$a = 3x,$$

$$x = \tfrac{1}{3}a.$$

c) What is the probability that the particle is in the region $0 \le x \le \frac{1}{2}a$?

We simply use the standard formula for the probability, namely

$$P\left(0 < x < \tfrac{1}{2}a\right) = \int_0^{\frac{1}{2}a} \left|\psi(x)\right|^2 dx = 12a^{-4}\int_0^{\frac{1}{2}a} x(a-x)^2\,dx = 12a^{-4}\int_0^{\frac{1}{2}a}\left(a^2x - 2ax^2 + x^3\right)dx$$

$$= 12a^{-4}\left(\tfrac{1}{2}a^2x^2 - \tfrac{2}{3}ax^3 + \tfrac{1}{4}x^4\right)\Big|_0^{\frac{1}{2}a} = a^{-4}\left(6a^2x^2 - 8ax^3 + 3x^4\right)\Big|_0^{\frac{1}{2}a}$$

$$= a^{-4}\left(\tfrac{3}{2}a^4 - a^4 + \tfrac{3}{16}a^4\right) = \tfrac{1}{2} + \tfrac{3}{16} = \tfrac{11}{16} = 68.75\%.$$

Part V: **Computation, New Material**

Choose three of the following four questions. Each question is worth 20 points. Type **only your answers to each part** into the essay box provided.

32. The isotope ^{230}Th is rare but does occur naturally. It has a half-life of $t_{1/2}$ = 75,400 years. A small table of isotopes is given to help you with this problem.

Z	Sym.	*A*	Mass (u)
88	Ra	226	226.02540
		230	230.03706
90	Th	226	226.02490
		230	230.03313
		234	234.04359
92	U	230	230.03394
		234	234.04095

a) What is the decay λ in y^{-1}?

We simply use the formula $\lambda t_{1/2} = \ln 2$ to find

$$\lambda = \frac{\ln 2}{t_{1/2}} = \frac{0.693}{7.54\times10^{4}\text{ y}} = 9.193\times10^{-6}\text{ y}^{-1}.$$

b) If you started with a mole (6.022×10^{23}) of ^{230}Th atoms, how many years would it take to reduce this to just one atom? Since most rocks are much older than this, clearly the ^{230}Th could not have been there all along.

We use the formula $N = N_0 e^{-\lambda t}$, where N = 1 and $N_0 = N_A$. We therefore have

$$e^{\lambda t} = \frac{N_0}{N} = \frac{N_A}{1} = 6.022\times10^{23},$$
$$\lambda t = \ln\left(6.022\times10^{23}\right) = 54.75,$$
$$t = \frac{54.75}{\lambda} = \frac{54.75}{9.193\times10^{-6}\text{ y}} = 5.956\times10^{6}\text{ y} = 5.956\text{ My}.$$

c) The ^{230}Th comes from the α-decay of some other isotope. What isotope is creating ^{230}Th?

Alpha decay decreases *A* by 4 and *Z* by 2. So the parent must have *Z* = 90+2 = 92 and *A* = 230+4 = 234, which would make it ^{234}U.

d) Calculate the *Q*-value for the decay that produces ^{230}Th. The mass of ^{4}He is m_{He} = 4.00260 u.

The formula for *Q* is just

$$Q = \left(M_P - M_D - M_{He}\right)c^2 = \left(234.04095 - 230.03313 - 4.00260\right)\text{u}c^2$$
$$= 0.00522\left(931.49\text{ MeV}\right) = 4.86\text{ MeV}.$$

33. Listed at right are the names and some information about several stars. For each star, determine the missing information. The surface temperature of the sun is $T_\odot = 5778\ \text{K}$.

Name	$L\,(L_\odot)$	$R\,(R_\odot)$	T (K)
Proxima Centauri	**0.00182**	0.154	3,042
α-Centauri A	1.519	**1.49**	5,260
α-Centauri B	0.500	0.863	**5,230**
Sirius A	**25.6**	1.711	9,940
Sirius B	0.056	**0.0126**	25,000
Betelgeuse	126,000	764	**3,940**

We will be using over and over the formula for the luminosity compared to the Sun, which takes the form

$$\frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{T_\odot}\right)^4 .$$

This can be easily solved for the radius or the temperature

$$\frac{R}{R_\odot} = \left(\frac{L}{L_\odot}\right)^{1/2} \left(\frac{T}{T_\odot}\right)^{-2}, \quad \text{and} \quad \frac{T}{T_\odot} = \left(\frac{L}{L_\odot}\right)^{1/4} \left(\frac{R}{R_\odot}\right)^{-1/2} .$$

For Proxima Centauri and Sirius B, we just use the straightforward formula:

$$\frac{L(\text{Proxima})}{L_\odot} = (0.154)^2 \left(\frac{3042\ \text{K}}{5778\ \text{K}}\right)^4 = 0.00182,$$

$$\frac{L(\text{Sirius A})}{L_\odot} = (1.711)^2 \left(\frac{9940\ \text{K}}{5778\ \text{K}}\right)^4 = 25.6.$$

For α-Centauri A and Sirius B, we use the formula for the radius:

$$\frac{R(\alpha\text{-Centauri A})}{R_\odot} = (1.519)^{1/2} \left(\frac{5260\ \text{K}}{5776\ \text{K}}\right)^{-2} = 1.49,$$

$$\frac{R(\text{Sirius B})}{R_\odot} = (0.056)^{1/2} \left(\frac{25000\ \text{K}}{5776\ \text{K}}\right)^{-2} = 0.0126.$$

We use the temperature formula for the remaining two cases

$$\frac{T(\alpha\text{-Centauri B})}{T_\odot} = (0.500)^{1/4} (0.863)^{-1/2} = 0.905,$$

$$\frac{T(\text{Betelgeuse})}{T_\odot} = \left(1.26\times10^5\right)^{1/4} (764)^{-1/2} = 0.681,$$

$$T(\alpha\text{-Centauri B}) = 0.905(5776\ \text{K}) = 5230\ \text{K},$$

$$T(\text{Betelgeuses}) = 0.682(5776\ \text{K}) = 3940\ \text{K}.$$

These numbers have all been included in the table above (typed in red).

34. The current temperature of the universe is T_0 = 2.7255 K. This question is about when the temperature was the temperature of boiling water, T = 373 K. This is lower than the temperature at matter-radiation equality, 8000 K.

a) What was the red-shift z at this time?

We use the formula $1+z = T/T_0 = a_0/a$, so we have

$$1+z = \frac{373\text{ K}}{2.7255\text{ K}} = 136.9\,.$$

So roughly z = 136.

b) What was the approximate age of the universe in My at this time? If you need it, g_{eff} = 3.36 at this time.

Because the temperature is *below* the transition temperature, we must use the matter-dominated formula. We therefore don't need g_{eff} (it was a distractor), so we simply have

$$t = \frac{17.2\text{ Gy}}{(1+z)^{3/2}} = \frac{17.2\text{ Gy}}{136.9^{1.5}} = 0.01074\text{ Gy} = 10.74\text{ My}.$$

c) The current density of the universe is around 0.251 atoms/m³. What was the density of atoms at this time?

The universe was smaller by a factor of 136.9 at the time, but this was in all three dimensions, so the *volume* was smaller by $136.9^3 = 2.566\times10^6$. The number of atoms per cubic meter will be *increased* by this factor, so we have

$$n = n_0(1+z)^3 = \left(0.251\text{ m}^{-3}\right)136.9^3 = 6.44\times10^5\text{ m}^{-3}$$

d) What was the ideal gas pressure from these atoms at this time?

This density is incredibly low by any normal standards, so we can find the pressure by using the ideal gas law, $P = k_BnT$:

$$\begin{aligned}P = k_BnT &= \left(1.3806\times10^{-23}\text{ J/K}\right)\left(6.44\times10^5\text{ m}^{-3}\right)\left(373\text{ K}\right)\\ &= 3.316\times10^{-15}\text{ J/m}^3 = 3.316\times10^{-15}\text{ N/m}^2 = 3.316\times10^{-15}\text{ Pa}\,.\end{aligned}$$

This is a very low gas pressure. The photon pressure can be shown to be immensely larger, but the photons are decoupled from the atoms.

35. In the book *Dragon's Egg*, by Robert L. Forward, a group of creatures called *cheela* live on the surface of a neutron star of mass $0.510 M_\odot$ (where $M_\odot = 1.989\times10^{30}$ kg), which has a radius of 20.3 km.

a) Suppose the cheela on the surface send a signal to humans who are far from the neutron star at a wavelength of 483 nm. At what wavelength will the humans detect it?

The wavelength will experience gravitational red-shift as it leaves the neutron star and reaches the humans. The formula is

$$\lambda = \frac{\lambda_0}{\sqrt{1-\dfrac{2GM}{c^2 r}}} = \frac{\lambda_0}{\sqrt{1-\dfrac{2\left(6.674\times10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}\right)(0.510)\left(1.989\times10^{30}\ \text{kg}\right)}{\left(2.998\times10^8\ \text{m/s}\right)^2\left(2.03\times10^4\ \text{m}\right)}}} = \frac{483\ \text{nm}}{\sqrt{1-0.0742}}$$

$$= 465\ \text{nm}.$$

b) Most of the story takes place over the course of approximately 30.0 days, as measured by the humans. How much time will pass according to the cheela?

The cheela will experience a proper time τ that is decreased due to the immense gravity well they live in. Fortunately, it is almost the exact same formula as before, so we have

$$\tau = t\sqrt{1-\frac{2GM}{c^2 r}} = (30.0\ \text{d})\sqrt{1-0.0742} = 28.9\ \text{d}.$$

However, as explained in the book, they actually experience time a million times faster than we do, so this turns into tens of thousands of years for them, enough for their civilization to rise from barbarism to much more advanced than we are.

c) Suppose that due to a natural disaster, the neutron star were to collapse to a black hole. What would be the resulting Schwarzschild radius of the black hole?

We simply use the formula for the Schwarzschild radius, namely

$$R_s = \frac{2GM}{c^2} = \frac{2\left(6.674\times10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}\right)(0.510)\left(1.989\times10^{30}\ \text{kg}\right)}{\left(2.998\times10^8\ \text{m/s}\right)^2} = 1507\ \text{m} = 1.507\ \text{km}.$$