Solutions to Problems 3a

1. [25] Show explicitly that eq. (3.21) with (3.22) satisfies the free Dirac equation, (3.5). Then repeat with eq. (3.23) with (3.24).

We will work with the right side and try to show that it equals the left side. Noting that $\partial_j \exp(i\mathbf{p}\cdot\mathbf{x}) = ip_j \exp(i\mathbf{p}\cdot\mathbf{x})$, we see that will be $\nabla \exp(i\mathbf{p}\cdot\mathbf{x}) = i\mathbf{p}\exp(i\mathbf{p}\cdot\mathbf{x})$, so we have

$$H\Psi \equiv (-i\boldsymbol{\alpha}\cdot\nabla + m\beta)\Psi = (-i\boldsymbol{\alpha}\cdot\nabla + m\beta)ue^{-ip\cdot x} = (-i\boldsymbol{\alpha}\cdot\nabla + m\beta)ue^{i\mathbf{p}\cdot\mathbf{x}-iEt} = (\boldsymbol{\alpha}\cdot\mathbf{p} + m\beta)ue^{i\mathbf{p}\cdot\mathbf{x}-iEt}$$

$$= \begin{pmatrix} \mathbf{p}\cdot\boldsymbol{\sigma} & -m \\ -m & -\mathbf{p}\cdot\boldsymbol{\sigma} \end{pmatrix} \begin{pmatrix} \sqrt{E\pm p}\,\psi_\pm \\ -\sqrt{E\mp p}\,\psi_\pm \end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{x}-iEt} = \begin{pmatrix} p\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\psi_\pm\sqrt{E\pm p} + m\sqrt{E\mp p}\,\psi_\pm \\ p\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\psi_\pm\sqrt{E\mp p} - m\sqrt{E\pm p}\,\psi_\pm \end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{x}-iEt}$$

$$= \begin{pmatrix} \left[\pm p\sqrt{E\pm p} + m\sqrt{E\mp p}\right]\psi_\pm \\ \left[\pm p\sqrt{E\mp p} - m\sqrt{E\pm p}\right]\psi_\pm \end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{x}-iEt}.$$

We are trying to equate this to

$$i\partial_0\Psi \equiv i\partial_0 ue^{i\mathbf{p}\cdot\mathbf{x}-iEt} = Eue^{i\mathbf{p}\cdot\mathbf{x}-iEt} = \begin{pmatrix} E\sqrt{E\pm p}\,\psi_\pm \\ -E\sqrt{E\mp p}\,\psi_\pm \end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{x}-iEt}.$$

Setting the two expressions equal to each other, and canceling the common exponentials, we are trying to prove

$$\begin{pmatrix} E\sqrt{E\pm p}\,\psi_\pm \\ -E\sqrt{E\mp p}\,\psi_\pm \end{pmatrix} = \begin{pmatrix} \left[\pm p\sqrt{E\pm p} + m\sqrt{E\mp p}\right]\psi_\pm \\ \left[\pm p\sqrt{E\mp p} - m\sqrt{E\pm p}\right]\psi_\pm \end{pmatrix},$$

$$E\sqrt{E\pm p} = \pm p\sqrt{E\pm p} + m\sqrt{E\mp p} \quad \text{and} \quad -E\sqrt{E\mp p} = \pm p\sqrt{E\mp p} - m\sqrt{E\pm p},$$

$$(E\mp p)\sqrt{E\pm p} = m\sqrt{E\mp p} \quad \text{and} \quad -(E\pm p)\sqrt{E\mp p} = -m\sqrt{E\pm p},$$

$$\sqrt{E\mp p}\sqrt{E\pm p} = m \quad \text{and} \quad -\sqrt{E\pm p}\sqrt{E\mp p} = -m,$$

$$\sqrt{E^2-p^2} = m \quad \text{and} \quad -\sqrt{E^2-p^2} = -m.$$

These equations are trivial to prove, so we are done with this case.

We now move on to the negative energy solutions. We have

$$H\Psi \equiv (-i\boldsymbol{\alpha}\cdot\nabla + m\beta)\Psi = (-i\boldsymbol{\alpha}\cdot\nabla + m\beta)ve^{ip\cdot x} = (-i\boldsymbol{\alpha}\cdot\nabla + m\beta)ve^{-i\mathbf{p}\cdot\mathbf{x}+iEt} = (-\boldsymbol{\alpha}\cdot\mathbf{p} + m\beta)ve^{-i\mathbf{p}\cdot\mathbf{x}+iEt}$$

$$= \begin{pmatrix} -\mathbf{p}\cdot\boldsymbol{\sigma} & -m \\ -m & \mathbf{p}\cdot\boldsymbol{\sigma} \end{pmatrix} \begin{pmatrix} \sqrt{E\mp p}\,\psi_\mp \\ \sqrt{E\pm p}\,\psi_\mp \end{pmatrix} e^{-i\mathbf{p}\cdot\mathbf{x}+iEt} = \begin{pmatrix} -p\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\psi_\mp\sqrt{E\mp p} - m\sqrt{E\pm p}\,\psi_\mp \\ p\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\psi_\mp\sqrt{E\pm p} - m\sqrt{E\mp p}\,\psi_\mp \end{pmatrix} e^{-i\mathbf{p}\cdot\mathbf{x}+iEt}$$

$$= \begin{pmatrix} \left[\pm p\sqrt{E\mp p} - m\sqrt{E\pm p}\right]\psi_\mp \\ \left[\mp p\sqrt{E\pm p} - m\sqrt{E\mp p}\right]\psi_\mp \end{pmatrix} e^{-i\mathbf{p}\cdot\mathbf{x}+iEt}.$$

We once again want to equate this to

$$i\partial_0\Psi \equiv i\partial_0 v e^{-i\mathbf{p}\cdot\mathbf{x}+iEt} = -Eve^{-i\mathbf{p}\cdot\mathbf{x}+iEt} = \begin{pmatrix} -E\sqrt{E\mp p}\,\psi_\mp \\ -E\sqrt{E\pm p}\,\psi_\mp \end{pmatrix} e^{-i\mathbf{p}\cdot\mathbf{x}+iEt} .$$

We therefore need

$$\begin{pmatrix} \left[\pm p\sqrt{E\mp p} - m\sqrt{E\pm p}\right]\psi_\mp \\ \left[\mp p\sqrt{E\pm p} - m\sqrt{E\mp p}\right]\psi_\mp \end{pmatrix} = \begin{pmatrix} -E\sqrt{E\mp p}\,\psi_\mp \\ -E\sqrt{E\pm p}\,\psi_\mp \end{pmatrix},$$

$$\pm p\sqrt{E\mp p} - m\sqrt{E\pm p} = -E\sqrt{E\mp p} \quad \text{and} \quad \mp p\sqrt{E\pm p} - m\sqrt{E\mp p} = -E\sqrt{E\pm p} ,$$

$$-m\sqrt{E\pm p} = -(E\pm p)\sqrt{E\mp p} \quad \text{and} \quad -m\sqrt{E\mp p} = -(E\mp p)\sqrt{E\pm p} ,$$

$$-m = -\sqrt{E\pm p}\sqrt{E\mp p} \quad \text{and} \quad -m = -\sqrt{E\mp p}\sqrt{E\pm p} ,$$

$$-m = -\sqrt{E^2-p^2} .$$

Again, the final line is trivial.

3. [20] Let Ψ be any four-component spinor. Consider the four quantities

$$S = \bar\Psi\Psi , \quad P = \bar\Psi\gamma_5\Psi , \quad V^\mu = \bar\Psi\gamma^\mu\Psi , \quad A^\mu = \bar\Psi\gamma^\mu\gamma_5\Psi .$$

(a) [5] Which of these are real, and which are pure imaginary?

Simply using the standard rules, we take the complex conjugate of each, keeping in mine (when necessary) that γ_5 anti-commutes with the others. So we have

$$S^* = \bar\Psi\Psi = S , \quad P^* = \bar\Psi\bar\gamma_5\Psi = -\bar\Psi\bar\gamma_5\Psi = -P ,$$

$$\left(V^\mu\right)^* = \bar\Psi\bar\gamma^\mu\Psi = \bar\Psi\gamma^\mu\Psi = V^\mu , \quad \left(A^\mu\right)^* = \bar\Psi\bar\gamma_5\bar\gamma^\mu\Psi = -\bar\Psi\gamma_5\gamma^\mu\Psi = \bar\Psi\gamma^\mu\gamma_5\Psi = A^\mu .$$

Hence only P is imaginary, the rest are real.

(b) [8] A *true scalar* is a quantity that is unchanged under parity. A *pseudoscalar* is a quantity that goes to minus itself under parity. A *true vector*, under parity, has its time component stay the same, while its vector part changes sign. An *axial vector*, under parity, has its time component change sign, while its space component stays the same. Classify these four quantities into the corresponding categories.

Under parity, these quantities become

$$\mathcal{P}(S) = \bar\Psi\gamma^0\gamma^0\Psi = \bar\Psi\Psi = S , \quad \mathcal{P}(P) = \bar\Psi\gamma^0\gamma_5\gamma^0\Psi = -\bar\Psi\gamma^0\gamma^0\gamma_5\Psi = -\bar\Psi\gamma_5\Psi = -P ,$$

$$\mathcal{P}\left(V^\mu\right) = \bar\Psi\gamma^0\gamma^\mu\gamma^0\Psi = \begin{cases} \bar\Psi\gamma^0\gamma^0\gamma^\mu\Psi & \text{if } \mu = 0 \\ -\bar\Psi\gamma^0\gamma^0\gamma^\mu\Psi & \text{if } \mu \neq 0 \end{cases} = \begin{cases} \bar\Psi\gamma^\mu\Psi & \text{if } \mu = 0 \\ -\bar\Psi\gamma^\mu\Psi & \text{if } \mu \neq 0 \end{cases} = \begin{cases} V^\mu & \text{if } \mu = 0 , \\ -V^\mu & \text{if } \mu \neq 0 , \end{cases}$$

$$\mathcal{P}\left(A^\mu\right) = \bar\Psi\gamma^0\gamma^\mu\gamma_5\gamma^0\Psi = -\bar\Psi\gamma^0\gamma^\mu\gamma^0\gamma_5\Psi = \begin{cases} -\bar\Psi\gamma^0\gamma^0\gamma^\mu\gamma_5\Psi & \text{if } \mu = 0 \\ \bar\Psi\gamma^0\gamma^0\gamma^\mu\gamma_5\Psi & \text{if } \mu \neq 0 \end{cases} = \begin{cases} -A^\mu & \text{if } \mu = 0 , \\ A^\mu & \text{if } \mu \neq 0 . \end{cases}$$

This means that *S*, *P*, *V* and *A* are scalar, pseudoscalar, vector, and axial vector respectively, as suggested by their names.

(c) [7] Suppose that a neutrino only has components in its lower half, so that $\Psi_R = 0$ and $\Psi_L \neq 0$. Which of these four quantities would be non-zero for a neutrino?

Let *X* denote a completely generic non-zero number or expression. Keeping in mind that γ^μ is block off-diagonal and γ_5 is on-diagonal, we have

$$S = \bar{\Psi}\Psi = \Psi^\dagger\gamma^0\Psi = \begin{pmatrix} 0 & X \end{pmatrix}\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 \\ X \end{pmatrix} = \begin{pmatrix} 0 & X \end{pmatrix}\begin{pmatrix} X \\ 0 \end{pmatrix} = 0,$$

$$P = \bar{\Psi}\gamma_5\Psi = \Psi^\dagger\gamma^0\gamma_5\Psi = \begin{pmatrix} 0 & X \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 \\ X \end{pmatrix} = \begin{pmatrix} 0 & X \end{pmatrix}\begin{pmatrix} X \\ 0 \end{pmatrix} = 0,$$

$$V^\mu = \bar{\Psi}\gamma^\mu\Psi = \Psi^\dagger\gamma^0\gamma^\mu\Psi = \begin{pmatrix} 0 & X \end{pmatrix}\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 \\ X \end{pmatrix} = \begin{pmatrix} X & 0 \end{pmatrix}\begin{pmatrix} X \\ 0 \end{pmatrix} = X,$$

$$A^\mu = \bar{\Psi}\gamma^\mu\gamma_5\Psi = \Psi^\dagger\gamma^0\gamma^\mu\gamma_5\Psi = \begin{pmatrix} 0 & X \end{pmatrix}\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}\begin{pmatrix} 0 \\ X \end{pmatrix} = \cdots = X.$$

Hence the vector and axial vector are allowed, but not the scalar or pseudoscalar.

4. [5] Suppose that a neutrino only has components in its lower half, so that $\Psi_R = 0$ and $\Psi_L \neq 0$. Which of the following seven symmetries carry the lower half to the lower half? Those that do not cannot be symmetries of neutrinos:
$\mathcal{C}$, $\mathcal{P}$, $\mathcal{T}$, $\mathcal{CP}$, $\mathcal{PT}$, $\mathcal{CT}$, $\mathcal{CPT}$.

Complex conjugation leaves the lower half in the lower half, so we can ignore its effects. We now work out what happens in each case.

$$\mathcal{C}\Psi = C\Psi^* = i\gamma^2\Psi^* = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 \\ X \end{pmatrix} = \begin{pmatrix} X \\ 0 \end{pmatrix},$$

$$\mathcal{P}\Psi = P\Psi = \gamma^0\Psi = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 \\ X \end{pmatrix} = \begin{pmatrix} X \\ 0 \end{pmatrix},$$

$$\mathcal{T}\Psi = T\Psi^* = -\Sigma^2\Psi = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}\begin{pmatrix} 0 \\ X \end{pmatrix} = \begin{pmatrix} 0 \\ X \end{pmatrix}.$$

Hence we see that charge conjugation and parity switch things, while time reversal does not. From this it isn't hard to conclude that, for example, $\mathcal{CP}$ switches things twice, while $\mathcal{PT}$ and $\mathcal{CT}$ cause only one switch, and the triple combination $\mathcal{CPT}$ also switches things twice. From this we conclude that only $\mathcal{T}$, $\mathcal{CP}$ and $\mathcal{CPT}$ are possible symmetries of neutrinos.