Solutions to Problems 6a

1. Simplify $\sum_s \bar{u}(p,s) M u(p,s)$ for the matrices $M = 1, \gamma_5, \gamma^\mu, \gamma_5\gamma^\mu$ and $\gamma^\mu\gamma^\nu$.

The trick is to simply write this as a trace, then we have

$$\sum_s \bar{u}(p,s) M u(p,s) = \sum_s \mathrm{Tr}\left[\bar{u}(p,s) M u(p,s)\right] = \sum_s \mathrm{Tr}\left[M u(p,s)\bar{u}(p,s)\right] = \mathrm{Tr}\left[M\left(\not{p}+m\right)\right].$$

We now simply work this out for each of the cases we have, keeping in mind that only even numbers of Dirac matrices (not counting γ_5's) contribute. So we have

$$\sum_s \bar{u}(p,s) 1 u(p,s) = \mathrm{Tr}\left[1\left(\not{p}+m\right)\right] = \mathrm{Tr}(m) = 4m\,,$$

$$\sum_s \bar{u}(p,s)\gamma_5 u(p,s) = \mathrm{Tr}\left[\gamma_5\left(\not{p}+m\right)\right] = m\mathrm{Tr}(\gamma_5) = 0\,,$$

$$\sum_s \bar{u}(p,s)\gamma^\mu u(p,s) = \mathrm{Tr}\left[\gamma^\mu\left(\not{p}+m\right)\right] = \mathrm{Tr}\left(\gamma^\mu \not{p}\right) = p_\nu \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4p_\nu g^{\mu\nu} = 4p^\nu\,,$$

$$\sum_s \bar{u}(p,s)\gamma_5\gamma^\mu u(p,s) = \mathrm{Tr}\left[\gamma_5\gamma^\mu\left(\not{p}+m\right)\right] = \mathrm{Tr}\left(\gamma_5\gamma^\mu \not{p}\right) = 0\,,$$

$$\sum_s \bar{u}(p,s)\gamma^\mu\gamma^\nu u(p,s) = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\left(\not{p}+m\right)\right] = \mathrm{Tr}\left(\gamma^\mu\gamma^\nu m\right) = 4mg^{\mu\nu}\,.$$

2. If $i\mathcal{M} = a\left[\bar{u}(p,s)\left(\not{p}+\not{p}'\right)(1-\gamma_5)v(p',s')\right]$, where *a* is constant, simplify $\sum_{s,s'}\left|i\mathcal{M}\right|^2$ as much as possible. Assume the mass associated with *p* is *m*, so $p^2 = m^2$, and the mass associated with p' is 0.

The first step is to simplify the expression as much as possible before proceeding. We note that $\not{p}$ is right next to $\bar{u}$, so we can immediately simplify $\bar{u}\not{p} = \bar{u}m$. Unfortunately, the $\not{p}'$ is not adjacent to v', but we can take advantage of the anti-commutation with γ_5 to rewrite this term as

$$\not{p}'(1-\gamma_5)v(p',s') = (1+\gamma_5)\not{p}'v(p',s') = 0\,.$$

Hence the whole expression simplifies to

$$i\mathcal{M} = a\left[\bar{u}(p,s)\not{p}(1-\gamma_5)v(p',s')\right] = ma\left[\bar{u}(p,s)(1-\gamma_5)v(p',s')\right]$$

The complex conjugate of this expression is

$$\left(i\mathcal{M}\right)^* = a^* m\left[\bar{v}(p',s')(1+\gamma_5)u(p,s)\right]$$

Multiplying this by the previous expression, we have

$$\left|i\mathcal{M}\right|^2 = m^2 aa^*\left[\bar{u}(p,s)(1-\gamma_5)v(p',s')\bar{v}(p',s')(1+\gamma_5)u(p,s)\right].$$

We have pushed together the two factors in anticipation of rewriting it as a trace.

We now sum over spins and rewrite the expression as a trace. We have

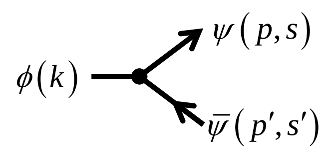
$$\begin{aligned}\sum_{s,s'}\left|i\mathcal{M}\right|^2 &= m^2 a a^* \sum_{s,s'}\left[\bar{u}(p,s)(1-\gamma_5)v(p',s')\bar{v}(p',s')(1+\gamma_5)u(p,s)\right] \\ &= m^2 a a^* \sum_{s,s'}\mathrm{Tr}\left[\bar{u}(p,s)(1-\gamma_5)v(p',s')\bar{v}(p',s')(1+\gamma_5)u(p,s)\right] \\ &= m^2 a a^* \sum_{s,s'}\mathrm{Tr}\left[u(p,s)\bar{u}(p,s)(1-\gamma_5)v(p',s')\bar{v}(p',s')(1+\gamma_5)\right] \\ &= m^2 a a^* \mathrm{Tr}\left[(\not p + m)(1-\gamma_5)(\not p' - 0)(1+\gamma_5)\right].\end{aligned}$$

We now take advantage of the fact that γ_5 anti-commutes with $\not p'$ to rewrite this as

$$\begin{aligned}\sum_{s,s'}\left|i\mathcal{M}\right|^2 &= m^2 a a^* \mathrm{Tr}\left[(\not p + m)\not p'(1+\gamma_5)(1+\gamma_5)\right] = m^2 a a^* \mathrm{Tr}\left[(\not p + m)\not p'\left(1+2\gamma_5+\gamma_5^2\right)\right] \\ &= 2m^2 a a^* \mathrm{Tr}\left[(\not p + m)\not p'(1+\gamma_5)\right] = 2m^2 a a^* \mathrm{Tr}\left[\not p \not p' + \not p \not p' \gamma_5\right] = 8m^2 a a^* (p\cdot p').\end{aligned}$$

That's about as simple as we can make it.

4. Calculate the decay rate $\phi \to \psi\bar{\psi}$ if we have scalar instead of pseudoscalar couplings.



The diagram is identical to the one in Fig. 6-4, but the rules are different, and the amplitude is $i\mathcal{M} = -ig(\bar{u}v')$. We therefore have

$$\left|i\mathcal{M}\right|^2 = (-ig)(ig)(\bar{u}v')(\bar{v}'u) = g^2(\bar{u}v'\bar{v}'u)$$

Summing on final state spins and introducing a trace in the usual way, we have

$$\begin{aligned}\sum_{s,s'}\left|i\mathcal{M}\right|^2 &= g^2\sum_{s,s'}\mathrm{Tr}(\bar{u}v'\bar{v}'u) = g^2\sum_{s,s'}\mathrm{Tr}(v'\bar{v}'u\bar{u}) = g^2\mathrm{Tr}\left[(\not p' - m)(\not p + m)\right] = g^2\mathrm{Tr}\left(\not p' \not p - m^2\right) \\ &= 4g^2\left(p\cdot p' - m^2\right).\end{aligned}$$

It is not hard to see that

$$\begin{aligned}M^2 = k^2 = (p+p')^2 = p^2 + p'^2 + 2p\cdot p' = 2m^2 + 2p\cdot p', \\ p\cdot p' = \tfrac{1}{2}M^2 - 2m^2.\end{aligned}$$

We therefore have

$$\sum_{s,s'}\left|i\mathcal{M}\right|^2 = 4g^2\left(\tfrac{1}{2}M^2 - m^2 - m^2\right) = 2g^2\left(M^2 - 4m^2\right).$$

We then use standard equations to carry us to the final decay rate, namely

$$\Gamma = \frac{D}{2M} = \frac{1}{2M}\frac{p}{16\pi^2 E_{cm}}\int\sum_{s,s'}\left|i\mathcal{M}\right|^2 d\Omega = \frac{2g^2 p}{32\pi^2 M^2}(4\pi)\left(M^2 - 4m^2\right) = \frac{g^2 p}{4\pi M^2}\left(M^2 - 4m^2\right).$$

The initial energy E is split evenly between the two final particles, so they each have energy $E = \tfrac{1}{2}M$, and hence momentum $p = \sqrt{\tfrac{1}{4}M^2 - m^2} = \tfrac{1}{2}\sqrt{M^2 - 4m^2}$. Putting it all together, we have

$$\Gamma = \frac{g^2}{8\pi M^2}\left(M^2 - 4m^2\right)^{3/2} .$$