Solutions to Problems 7c

10. Consider again the Feynman amplitude for $e^-(p)\gamma(q)\to e^-(p')\gamma(q')$. Show that if you replace the polarization of one of the photons (either one – your choice) with its corresponding momentum, the Feynman amplitude becomes exactly zero. Do not treat the electron as massless.

A moment of thought is worth using before we plunge into the calculation. We need to decide which photon we are going to make the substitution for. We'll pick the initial photon with momentum *q*. For this reason, it is advisable to write the intermediate momenta in terms of *q*, since this will give the greatest chance for simplification. Furthermore, since we are trying to prove something, we write the propagators in the "proof" form. Hence our Feynman amplitude will take the form

$$i\mathcal{M}=(ie)^2\,\varepsilon_\mu\varepsilon_\nu'^*\left[\bar{u}'\gamma^\nu\frac{i}{\not{p}+\not{q}-m}\gamma^\mu u+\bar{u}'\gamma^\mu\frac{i}{\not{p}'-\not{q}-m}\gamma^\nu u\right].$$

We now make the substitution $\varepsilon_\mu\to q_\mu$ and simplify this to

$$i\mathcal{M}=-ie^2\varepsilon_\nu'^*\left[\bar{u}'\gamma^\nu\frac{1}{\not{p}+\not{q}-m}\not{q}u+\bar{u}'\not{q}\frac{1}{\not{p}'-\not{q}-m}\gamma^\nu u\right].$$

For the first term, we note that $\not{p}u=mu$, so we can rewrite $\not{q}u=(\not{q}+\not{p}-m)u$. For the second term, we note that $\bar{u}'\not{p}'=\bar{u}'m$, so we can rewrite $\bar{u}'\not{q}=\bar{u}'(\not{q}-\not{p}'+m)$. Substituting these expressions in, we find

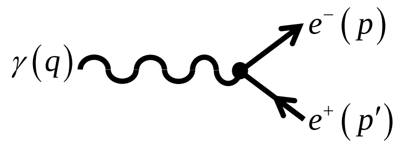
$$\begin{aligned}i\mathcal{M}&=-ie^2\varepsilon_\nu'^*\left[\bar{u}'\gamma^\nu\frac{1}{\not{p}+\not{q}-m}(\not{q}+\not{p}-m)u+\bar{u}'(\not{q}-\not{p}'+m)\frac{1}{\not{p}'-\not{q}-m}\gamma^\nu u\right]\\&=-ie^2\varepsilon_\nu'^*\left[\bar{u}'\gamma^\nu(1)u+\bar{u}'(-1)\gamma^\nu u\right]=0.\end{aligned}$$

So we have proven it.

12. If the photon were massive, there would be *three* polarizations of the photon, and the sum over polarizations rule would be changed to

$$\sum_{\sigma=1}^{3} \varepsilon_\mu^*(\mathbf{q},\sigma)\,\varepsilon_\nu(\mathbf{q},\sigma) = -g_{\mu\nu} + q_\mu q_\nu / M^2 \,,$$

where *M* is the photon mass. Calculate the decay rate $\gamma \to e^+ e^-$, assuming no other Feynman rules change. Include the electron mass *m*.



There is only one Feynman diagram, sketched at right. According to our Feynman rules, the amplitude is

$$i\mathcal{M} = ie\varepsilon_\mu\left(\bar{u}\gamma^\mu v'\right),$$

$$\left|i\mathcal{M}\right|^2 = e^2 \varepsilon_\mu \varepsilon_\nu^* \left(\bar{u}\gamma^\mu v'\right)\left(\bar{v}'\gamma^\nu u\right).$$

We average this over initial polarizations and sum over final spins, to give us

$$\begin{aligned}
\tfrac{1}{3}\sum\left|i\mathcal{M}\right|^2 &= \tfrac{1}{3}e^2\left(-g_{\mu\nu} + q_\mu q_\nu / M^2\right)\mathrm{Tr}\left[\left(\not{p}+m\right)\gamma^\mu\left(\not{p}'-m\right)\gamma^\nu\right] \\
&= -\tfrac{1}{3}e^2\mathrm{Tr}\left[\left(\not{p}+m\right)\gamma^\mu\left(\not{p}'-m\right)\gamma_\mu\right] + \frac{e^2}{3M^2}\mathrm{Tr}\left[\left(\not{p}+m\right)\not{q}\left(\not{p}'-m\right)\not{q}\right] \\
&= -\tfrac{1}{3}e^2\mathrm{Tr}\left(\not{p}\gamma^\mu\not{p}'\gamma_\mu - m^2\gamma^\mu\gamma_\mu\right) + \frac{e^2}{3M^2}\mathrm{Tr}\left(\not{p}\not{q}\not{p}'\not{q} - m^2\not{q}^2\right) \\
&= \tfrac{1}{3}e^2\mathrm{Tr}\left(2\not{p}\not{p}' + 4m^2\right) + \frac{e^2}{3M^2}\mathrm{Tr}\left(\not{p}\not{q}\not{p}'\not{q} - m^2M^2\right) \\
&= \tfrac{8}{3}e^2\left(p\cdot p'\right) + \tfrac{16}{3}e^2m^2 + \frac{4e^2}{3M^2}\left[2\left(p\cdot q\right)\left(p'\cdot q\right) - \left(p\cdot p'\right)M^2 - m^2M^2\right] \\
&= \tfrac{4}{3}e^2\left(p+p'\right)^2 + \tfrac{8}{3}e^2m^2 + \frac{4e^2}{3M^2}\left[2\left(p\cdot q\right)\left(p'\cdot q\right) - \tfrac{1}{2}M^2\left(p+p'\right)^2\right]
\end{aligned}$$

We now note that $\left(p+p'\right)^2 = M^2$, and that since $q = \left(M,0,0,0\right)$ and this energy is split evenly between the final state particles, we will have $p\cdot q = p'\cdot q = \tfrac{1}{2}M^2$. This allows us to simplify the final expression to

$$\tfrac{1}{3}\sum\left|i\mathcal{M}\right|^2 = \tfrac{4}{3}e^2M^2 + \tfrac{8}{3}e^2m^2 + \frac{4e^2}{3M^2}\left[2\cdot\tfrac{1}{2}M^2\cdot\tfrac{1}{2}M^2 - \tfrac{1}{2}M^4\right] = \tfrac{4}{3}e^2\left(M^2 + 2m^2\right).$$

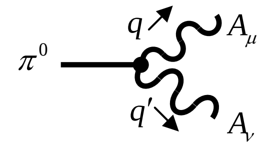
The decay rate is therefore

$$\Gamma = \frac{D}{2M} = \frac{1}{2M}\cdot\frac{|\mathbf{p}|}{16\pi^2 M}\int \tfrac{1}{3}\sum\left|i\mathcal{M}\right|^2 d\Omega = \frac{4e^2|\mathbf{p}|}{3\cdot 32\pi^2 M^2}\int\left(M^2 + 2m^2\right)d\Omega = \tfrac{2}{3}\alpha|\mathbf{p}|\left(1 + 2m^2/M^2\right).$$

The final momentum is $|\mathbf{p}| = \sqrt{\left(\tfrac{1}{2}M\right)^2 - m^2}$, so substituting this in, we have

$$\Gamma = \tfrac{1}{3}\alpha\sqrt{M^2 - 4m^2}\left(1 + 2m^2/M^2\right).$$

13. The π^0 is not a fundamental particle, but made of quark/anti-quark pairs. It decays by the process $\pi^0 \to \gamma\gamma$. The coupling responsible is non-renormalizable, and is given in Fig. 7-11, where *g* is the coupling. Find the rate for this decay.



$$ig\varepsilon^{\mu\alpha\nu\beta}q_\alpha q'_\beta$$

Figure 7-11: The interaction responsible for π^0 decay.

There is only one diagram, identical with Fig. 7-11, so we won't redraw it. The amplitude is

$$i\mathcal{M} = ig\varepsilon^{\mu\alpha\nu\beta}q_\alpha q'_\beta \varepsilon^*_\mu \varepsilon'^*_\nu$$

We square this and then sum over the final polarizations, which yields

$$\begin{aligned}\sum\left|i\mathcal{M}\right|^2 &= \sum g^2\varepsilon^{\mu\alpha\nu\beta}q_\alpha q'_\beta \varepsilon^*_\mu \varepsilon'^*_\nu \varepsilon^{\rho\sigma\tau\lambda}q_\sigma q'_\lambda \varepsilon_\rho \varepsilon'_\tau = g^2 g_{\mu\rho}g_{\nu\tau}\varepsilon^{\mu\alpha\nu\beta}q_\alpha q'_\beta \varepsilon^{\rho\sigma\tau\lambda}q_\sigma q'_\lambda \\ &= g^2 q_\alpha q'_\beta q_\sigma q'_\lambda \varepsilon^{\mu\alpha\nu\beta}\varepsilon_{\mu\ \nu}^{\ \sigma\ \lambda} = 2g^2\left[\left(q\cdot q'\right)\left(q\cdot q'\right)-\left(q\cdot q\right)\left(q'\cdot q'\right)\right] = 2g^2\left(q\cdot q'\right)^2,\end{aligned}$$

where at the penultimate step we used the solution to problem 2.2c. We then note that

$$m_\pi^2 = \left(q+q'\right)^2 = q^2 + q'^2 + 2q\cdot q' = 2q\cdot q'.$$

We therefore have $\sum\left|i\mathcal{M}\right|^2 = \tfrac{1}{2}g^2 m_\pi^4$. This allows us to proceed to the decay rate. However, there is an extra factor of ½ because we have identical particles in the final state.

$$\Gamma = \frac{D}{2m_\pi} = \frac{1}{2m_\pi}\frac{|\mathbf{q}|}{16\pi^2 m_\pi}\cdot\frac{1}{2}\int\sum\left|i\mathcal{M}\right|^2 d\Omega = \frac{1}{64\pi^2 m_\pi}\cdot\frac{4\pi}{2}\tfrac{1}{2}g^2 m_\pi^4 = \frac{g^2 m_\pi^3}{64\pi}.$$