Solutions to Problems 8b

9. Work out $T_{1\to3}\left|B^*\right\rangle$ for all nine states in eq. (8.45). Then copy Fig. 8-3, and draw an arrow showing what happens to each of these states. For example, since $T_{1\to3}\left|\Xi^{*0}\right\rangle=\sqrt{3}\left|\Omega^-\right\rangle$, you would draw an arrow from $\left|\Xi^{*0}\right\rangle$ to $\left|\Omega^-\right\rangle$.

This is pretty straightforward. We have

$$T_{1\to3}\left|\Delta^-\right\rangle=T_{1\to3}\left|B^*_{222}\right\rangle=0\,,$$

$$T_{1\to3}\left|\Delta^0\right\rangle=\tfrac{1}{\sqrt3}T_{1\to3}\left(\left|B^*_{122}\right\rangle+\left|B^*_{212}\right\rangle+\left|B^*_{221}\right\rangle\right)=\tfrac{1}{\sqrt3}\left(\left|B^*_{322}\right\rangle+\left|B^*_{232}\right\rangle+\left|B^*_{223}\right\rangle\right)=\left|\Sigma^{*-}\right\rangle,$$

$$T_{1\to3}\left|\Delta^+\right\rangle=\tfrac{1}{\sqrt3}T_{1\to3}\left(\left|B^*_{112}\right\rangle+\left|B^*_{121}\right\rangle+\left|B^*_{211}\right\rangle\right)=\tfrac{1}{\sqrt3}\left(\left|B^*_{312}\right\rangle+\left|B^*_{132}\right\rangle+\left|B^*_{321}\right\rangle+\left|B^*_{123}\right\rangle+\left|B^*_{231}\right\rangle+\left|B^*_{213}\right\rangle\right)$$
$$=\sqrt2\left|\Sigma^{*0}\right\rangle,$$

$$T_{1\to3}\left|\Delta^{++}\right\rangle=T_{1\to3}\left|B^*_{111}\right\rangle=\left|B^*_{311}\right\rangle+\left|B^*_{131}\right\rangle+\left|B^*_{113}\right\rangle=\sqrt3\left|\Sigma^{*+}\right\rangle,$$

$$T_{1\to3}\left|\Sigma^{*-}\right\rangle=\tfrac{1}{\sqrt3}T_{1\to3}\left(\left|B^*_{223}\right\rangle+\left|B^*_{232}\right\rangle+\left|B^*_{322}\right\rangle\right)=0\,,$$

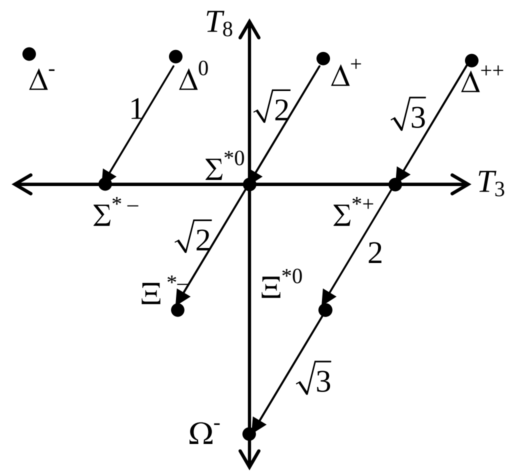
$$T_{1\to3}\left|\Sigma^{*0}\right\rangle=\tfrac{1}{\sqrt6}T_{1\to3}\left(\left|B^*_{123}\right\rangle+\left|B^*_{213}\right\rangle+\left|B^*_{132}\right\rangle+\left|B^*_{231}\right\rangle+\left|B^*_{312}\right\rangle+\left|B^*_{321}\right\rangle\right)$$
$$=\tfrac{1}{\sqrt6}\left(\left|B^*_{323}\right\rangle+\left|B^*_{233}\right\rangle+\left|B^*_{332}\right\rangle+\left|B^*_{233}\right\rangle+\left|B^*_{332}\right\rangle+\left|B^*_{323}\right\rangle\right)=\sqrt2\left|\Xi^{*1}\right\rangle,$$

$$T_{1\to3}\left|\Sigma^{*+}\right\rangle=\tfrac{1}{\sqrt3}T_{1\to3}\left(\left|B^*_{113}\right\rangle+\left|B^*_{131}\right\rangle+\left|B^*_{311}\right\rangle\right)=\tfrac{1}{\sqrt3}\left(\left|B^*_{313}\right\rangle+\left|B^*_{133}\right\rangle+\left|B^*_{331}\right\rangle+\left|B^*_{133}\right\rangle+\left|B^*_{331}\right\rangle+\left|B^*_{313}\right\rangle\right)$$
$$=2\left|\Xi^{*0}\right\rangle,$$

$$T_{1\to3}\left|\Xi^{*0}\right\rangle=\tfrac{1}{\sqrt3}T_{1\to3}\left(\left|B^*_{133}\right\rangle+\left|B^*_{313}\right\rangle+\left|B^*_{331}\right\rangle\right)=\tfrac{1}{\sqrt3}\left(\left|B^*_{333}\right\rangle+\left|B^*_{333}\right\rangle+\left|B^*_{333}\right\rangle\right)=\sqrt3\left|\Omega^-\right\rangle,$$

$$T_{1\to3}\left|\Xi^{*-}\right\rangle=\tfrac{1}{\sqrt3}T_{1\to3}\left(\left|B^*_{233}\right\rangle+\left|B^*_{323}\right\rangle+\left|B^*_{332}\right\rangle\right)=0\,.$$

You can also easily see that $T_{1\to3}\left|\Omega^-\right\rangle=0$. The relevant connections have been sketched at right. The operation always moves you down and to the left, and by exactly the same amount in every case. I have also added in the relevant factor in every case. Because there are no arrows on the objects on the lower left edge, these states vanish when acted on by $T_{1\to3}$.



11. Assume the masses of the baryons in the octet are given by

$$\left\langle B_i^j \middle| \mathcal{H} \middle| B_k^\ell \right\rangle = X\delta_k^i \delta_j^\ell + Y\left(T_8\right)_k^i \delta_j^\ell + Z\delta_k^i \left(T_8\right)_j^\ell ,$$

where *X*, *Y* and *Z* are constants.

(a) Find formulas for m_N, m_Λ, and m_Σ and m_Ξ in terms of *X*, *Y*, and *Z*.

Because isospin is a pretty good symmetry, we can use any member of an isospin multiplet to get the corresponding mass. For the proton, for example, we have

$$m_p = \left\langle p^+ \middle| \mathcal{H} \middle| p^+ \right\rangle = \left\langle B_1^3 \middle| \mathcal{H} \middle| B_1^3 \right\rangle = X\delta_1^1\delta_3^3 + Y\left(T_8\right)_1^1 \delta_3^3 + Z\delta_1^1\left(T_8\right)_3^3 = X + \tfrac{1}{2\sqrt{3}}Y - \tfrac{1}{\sqrt{3}}Z .$$

We similarly work out all the other combinations, though some of them are a little more complicated. We note that because both δ and T_8 are diagonal, we never need to consider "cross-terms" where the corresponding indices don't match. For example, for m_Λ, we write this as

$$\begin{aligned} m_\Lambda &= \left\langle \Lambda^0 \middle| \mathcal{H} \middle| \Lambda^0 \right\rangle = \tfrac{1}{6}\left\langle B_1^1 \middle| \mathcal{H} \middle| B_1^1 \right\rangle + \tfrac{1}{6}\left\langle B_2^2 \middle| \mathcal{H} \middle| B_2^2 \right\rangle + \tfrac{2}{3}\left\langle B_3^3 \middle| \mathcal{H} \middle| B_3^3 \right\rangle \\ &= \tfrac{1}{6}\left[X\delta_1^1\delta_1^1 + Y\left(T_8\right)_1^1 \delta_1^1 + Z\delta_1^1\left(T_8\right)_1^1 \right] + \tfrac{1}{6}\left[X\delta_2^2\delta_2^2 + Y\left(T_8\right)_2^2 \delta_2^2 + Z\delta_2^2\left(T_8\right)_2^2 \right] \\ &\quad + \tfrac{2}{3}\left[X\delta_3^3\delta_3^3 + Y\left(T_8\right)_3^3 \delta_3^3 + Z\delta_3^3\left(T_8\right)_3^3 \right] \\ &= \tfrac{1}{3}\left[X + \tfrac{1}{2\sqrt{3}}Y + \tfrac{1}{2\sqrt{3}}Z \right] + \tfrac{2}{3}\left[X - \tfrac{1}{\sqrt{3}}Y - \tfrac{1}{\sqrt{3}}Z \right] = X - \tfrac{1}{2\sqrt{3}}Y - \tfrac{1}{2\sqrt{3}}Z . \end{aligned}$$

By comparison, the other expressions are easy, since we can pick the particles to make the expressions as simple as possible. We have

$$m_\Sigma = \left\langle \Sigma^+ \middle| \mathcal{H} \middle| \Sigma^+ \right\rangle = \left\langle B_1^2 \middle| \mathcal{H} \middle| B_1^2 \right\rangle = X\delta_1^1\delta_2^2 + Y\left(T_8\right)_1^1 \delta_2^2 + Z\delta_1^1\left(T_8\right)_2^2 = X + \tfrac{1}{2\sqrt{3}}Y + \tfrac{1}{2\sqrt{3}}Z ,$$

$$m_\Xi = \left\langle \Xi^0 \middle| \mathcal{H} \middle| \Xi^0 \right\rangle = \left\langle B_3^2 \middle| \mathcal{H} \middle| B_3^2 \right\rangle = X\delta_3^3\delta_2^2 + Y\left(T_8\right)_3^3 \delta_2^2 + Z\delta_3^3\left(T_8\right)_2^2 = X - \tfrac{1}{\sqrt{3}}Y + \tfrac{1}{2\sqrt{3}}Z .$$

(b) Eliminate *X*, *Y*, and *Z* to show that $2m_N + 2m_\Xi = 3m_\Lambda + m_\Sigma$. Demonstrate that this works pretty well.

We have four equations in three unknowns, and hence should be able to eliminate all the variables. It is easiest just to write out the two sides of this equation and check that they come out the same. We have

$$2m_N + 2m_\Xi = 2\left(X + \tfrac{1}{2\sqrt{3}}Y - \tfrac{1}{\sqrt{3}}Z \right) + 2\left(X - \tfrac{1}{\sqrt{3}}Y + \tfrac{1}{2\sqrt{3}}Z \right) = 4X - \tfrac{1}{\sqrt{3}}Y - \tfrac{1}{\sqrt{3}}Z ,$$

$$3m_\Lambda + m_\Sigma = 3\left(X - \tfrac{1}{2\sqrt{3}}Y - \tfrac{1}{2\sqrt{3}}Z \right) + \left(X + \tfrac{1}{2\sqrt{3}}Y + \tfrac{1}{2\sqrt{3}}Z \right) = 4X - \tfrac{1}{\sqrt{3}}Y - \tfrac{1}{\sqrt{3}}Z .$$

These expressions are obviously equal. Now, if we use the average mass of each particle in each multiplet, we have

$$2m_N + 2m_\Xi = 2(939\text{ MeV}) + 2(1318\text{ MeV}) = 4514\text{ MeV},$$
$$3m_\Lambda + m_\Sigma = 3(1116\text{ MeV}) + (1193\text{ MeV}) = 4541\text{ MeV}.$$

These numbers are less than 1 percent apart, so this works pretty well.

(c) Since the mesons are also in an octet, we might think this implies $4m_K = 3m_\eta + m_\pi$ (where we used the fact that the kaons have the same masses as their anti-particles). Use this to "predict" the kaon mass, and show that this doesn't work very well. What went wrong? (hint – what do Hamiltonian matrix elements represent for bosons?) Fix the formula and show that it now works better.

Naively substituting the numbers in, we would have

$$4m_K = 4(496\text{ MeV}) = 1986\text{ MeV},$$
$$3m_\eta + m_\pi = 3(548\text{ MeV}) + (137\text{ MeV}) = 1781\text{ MeV}.$$

These numbers differ by about 11%, not very impressive. But then we remember that matrix elements for bosons represent the *squares* of their masses, so the correct relationship should be

$$4m_K^2 = 3m_\eta^2 + m_\pi^2.$$

Substituting the numbers in, and switching to GeV to make the numbers less intense, we have

$$4m_K^2 = 4(0.496\text{ GeV})^2 = 0.984\text{ GeV}^2,$$
$$3m_\eta + m_\pi = 3(0.548\text{ GeV})^2 + (0.137\text{ GeV})^2 = 0.920\text{ GeV}^2.$$

This is about 7% off, which is a little better. If we take the square root and treat it as a prediction for the kaon mass, it is only about 3.5% off, which isn't bad.