Physics 744 – Quantum Field Theory

Solution Set 2

1. **[5] Let x, y, z, and w be four independent four-vectors. We wish to form a scalar quantity *s* that is Lorentz invariant under proper Lorentz transformations and is linear in each of these four quantities, *i.e.*, it will contain expressions like *xyzw*, but we want to show explicitly how the indices can be put together.**
   (a) [3] What is the most general expression that can be formed of this type? There should be four linearly independent terms.

We need to write something like $s = x^\alpha y^\beta z^\gamma w^\delta$, but we need to get rid of all the spare indices. This can be done by contracting them together, for example, writing terms like $s = x^\alpha y_\alpha z^\gamma w_\gamma = (\mathbf{x}\cdot\mathbf{y})(\mathbf{z}\cdot\mathbf{w})$, and there will be three similar terms. We can also try to get rid of indices by contracting with the Levi-Civita tensor. Since this tensor is completely anti-symmetric, it doesn't matter which index we contract with which, so in summary the most general expression will look like

$$s = A(\mathbf{x}\cdot\mathbf{y})(\mathbf{z}\cdot\mathbf{w}) + B(\mathbf{x}\cdot\mathbf{z})(\mathbf{y}\cdot\mathbf{w}) + C(\mathbf{x}\cdot\mathbf{w})(\mathbf{z}\cdot\mathbf{y}) + D\varepsilon_{\alpha\beta\gamma\delta}x^\alpha y^\beta z^\gamma w^\delta$$

   (b) [2] A term is called a *true scalar* if it is invariant under parity, and a *pseudoscalar* if it changes sign under parity. Classify the four terms as scalars or pseudoscalars.

Under parity, the expression $\mathbf{x}\cdot\mathbf{y} = x^0 y^0 - \vec{x}\cdot\vec{y}$ remains unchanged, because the time part is unchanged and the space part is reversed. Hence the terms with coefficients *A*, *B*, and *C* are all true scalars. In contrast, if you look at $\varepsilon_{\alpha\beta\gamma\delta}x^\alpha y^\beta z^\gamma w^\delta$, it is clear that three of the indices must be space indices and one of them will be time, so that under parity it acquires three minus signs, for a net factor of negative one. Therefore the *D* term is a pseudoscalar.

2. **[15] In classical physics, if an object of mass *m* hits an object of identical mass, the two objects will head off at a 90 degree angle compared to each other. Consider an object of mass *m* moving at speed v_i and colliding elastically with another object of mass *m*. The two move off at identical speeds v_f at angles θ_1 and θ_2.**
   (a) [6] Write the four-momentum of all the incoming and outgoing particles, and write the conservation of four-momentum in components.

Let's work in a frame such that the initial particle is moving in the *x*-direction and the final particles are moving in the *xy*-plane. Then the four momentum of the particles will be:

initial: $\mathbf{p}_1 = m\gamma_i\left(1, v_i, 0, 0\right)$ $\mathbf{p}_2 = m\left(1, 0, 0, 0\right)$

final: $\mathbf{p}_1' = m\gamma_f\left(1, v_f\cos\theta_1, v_f\sin\theta_1, 0\right)$ $\mathbf{p}_2' = m\gamma_f\left(1, v_f\cos\theta_2, -v_f\sin\theta_2, 0\right)$

Conservation of four-momentum tells us $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'$. Ignoring the trivial *z*-component, and cancelling the common factor of *m*, we see that

$$\gamma_i + 1 = 2\gamma_f,$$
$$\gamma_i v_i = 2\gamma_f v_f \cos\theta_f,$$
$$0 = \gamma_f v_f\left(\cos\theta_1 - \cos\theta_2\right).$$

(b) [1] Show that $\theta_1 = \theta_2$.

This follows trivially from the third equation.

(c) [2] Find a formula for γ_f in terms of the initial velocity.

This follows directly from the first equation, $\gamma_f = \tfrac{1}{2}\left(1+\gamma_i\right)$. if we want it more explicit, we can write this as $\gamma_f = \tfrac{1}{2}\left(1 + 1\big/\sqrt{1-v_i^2}\right)$.

(d) [6] Show that the final angle is given by $\cos^2\theta = \left(\gamma_i+1\right)/\left(\gamma_i+3\right)$. Hence show that the outgoing particles are perpendicular in the non-relativistic limit. What happens in the ultrarelativistic limit?

Solving the only remaining equation, we have

$$\left(\cos\theta_f\right)^2 = \left(\frac{\gamma_i v_i}{2\gamma_f v_f}\right)^2 = \frac{\gamma_i^2 v_i^2}{4\gamma_f^2 v_f^2}.$$

From the definition of $\gamma = 1\big/\sqrt{1-v^2}$ it is easy to show that $\gamma^2\left(1-v^2\right) = 1$, which we rearrange as $\gamma^2 v^2 = \gamma^2 - 1$. Substituting, we find

$$\cos^2\theta_f = \frac{\gamma_i^2 v_i^2}{4\gamma_f^2 v_f^2} = \frac{\gamma_i^2 - 1}{4\left(\gamma_f^2 - 1\right)} = \frac{\gamma_i^2 - 1}{4\left[\frac{1}{4}\left(\gamma_i + 1\right)^2 - 1\right]} = \frac{\gamma_i^2 - 1}{\gamma_i^2 + 2\gamma_i - 3} = \frac{\left(\gamma_i + 1\right)\left(\gamma_i - 1\right)}{\left(\gamma_i + 3\right)\left(\gamma_i - 1\right)} = \frac{\gamma_i + 1}{\gamma_i + 3}$$

In the non-relativistic limit, we have $\gamma_i = 1$ and therefore $\cos^2\theta_f = \frac{1}{2}$, $\cos^2\theta_f = \frac{1}{2}$, $\cos\theta_f = \frac{1}{\sqrt{2}}$, corresponding to an angle of 45 degrees, and hence the outgoing particles are perpendicular. In the relativistic limit, $\gamma_i = \infty$ and therefore $\cos^2\theta_f = 1$, and both particles go forward, with an opening angle approaching zero.

3. [10] A *Z*-particle (mass m_Z) at rest decays to an electron (mass effectively zero) with energy E_1, a positron (also massless) with energy E_2 moving at an angle θ compared to it, and an invisible *X* particle of unknown mass. Find a formula for the unknown mass m_X^2.

We first denote the various momenta by $\mathbf{p}_Z$, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_X$. Conservation of four-momentum tells us that

$$\mathbf{p}_Z = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_X .$$

Now, we know everything about the *Z*'s momentum, and we know a great deal about the momentum of each of the two electrons. The *X* we know nothing about, but we do want its mass. Fortunately, squaring $\mathbf{p}_X$ will give us the mass, without exploring the rest of our ignorance. We therefore solve this equation for $\mathbf{p}_X$ and then square the resulting expression.

$$\begin{aligned}
\mathbf{p}_X &= \mathbf{p}_Z - \mathbf{p}_1 - \mathbf{p}_2 , \\
\mathbf{p}_X^2 &= \left(\mathbf{p}_Z - \mathbf{p}_1 - \mathbf{p}_2\right)^2 , \\
m_X^2 &= \mathbf{p}_Z^2 + \mathbf{p}_1^2 + \mathbf{p}_2^2 - 2\mathbf{p}_1 \cdot \mathbf{p}_Z - 2\mathbf{p}_2 \cdot \mathbf{p}_z + 2\mathbf{p}_1 \cdot \mathbf{p}_2 \\
&= m_Z^2 + 0 + 0 - 2\mathbf{p}_1 \cdot \mathbf{p}_Z - 2\mathbf{p}_2 \cdot \mathbf{p}_z + 2\mathbf{p}_1 \cdot \mathbf{p}_2 .
\end{aligned}$$

We have treated the electron and positron as effectively massless. The *Z* has no momentum (it is at rest), and therefore the dot product of its four-momentum with the electron or positron is $\mathbf{p}_Z \cdot \mathbf{p}_1 = E_Z E_1 - \vec{p}_Z \cdot \vec{p}_1 = m_Z E_1$ and $\mathbf{p}_Z \cdot \mathbf{p}_2 = m_Z E_2$. Finally, we have

$$\mathbf{p}_1 \cdot \mathbf{p}_2 = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2 = E_1 E_2 - p_1 p_2 \cos\theta = E_1 E_2 - E_1 E_2 \cos\theta .$$

Substituting everything in, we have

$$m_X^2 = m_Z^2 - 2m_Z E_1 - 2m_Z E_2 + 2E_1 E_2 \left(1 - \cos\theta\right).$$