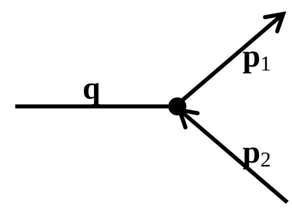
Physics 744 - Field Theory

Solution Set 8

1. In the $\bar{\psi}\psi\phi$ theory, calculate the unpolarized rate for the decay $\phi \to \bar{\psi}\psi$ assuming $M > 2m$ for the case of (a) scalar coupling and (b) pseudoscalar coupling.



There is only one relevant Feynman diagram, sketched above. The corresponding Feynman amplitude will be

$$i\mathcal{M} = -ig\left(\bar{u}_1 v_2\right) \quad \text{or} \quad i\mathcal{M} = g\left(\bar{u}_1\gamma_5 v_2\right)$$

We now want to multiply by the complex conjugate, which is the same as barring. Recall when you bar something, you reverse the order of the arguments, and $\bar{\gamma}_5 = -\gamma_5$. We therefore have

$$\left|i\mathcal{M}\right|^2 = g^2\left(\bar{u}_1 v_2\right)\left(\bar{v}_2 u_1\right) \quad \text{or} \quad \left|i\mathcal{M}\right|^2 = -g^2\left(\bar{u}_1\gamma_5 v_2\right)\left(\bar{v}_2\gamma_5 u_1\right)$$

We now want to sum over all final state spins (and average over incoming spins, except there aren't any). We use the usual trick of writing it as a trace so we can use our identities on the sums of spins. We have:

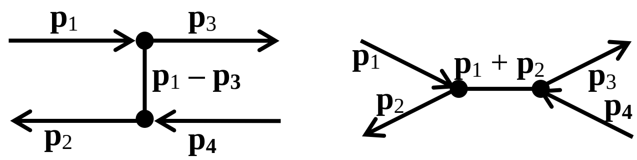
$$\begin{aligned}
\sum_{s_1,s_2}\left|i\mathcal{M}\right|^2 &= g^2\sum_{s_1,s_2}\mathrm{Tr}\left(\bar{u}_1 v_2\bar{v}_2 u_1\right) = g^2\sum_{s_1,s_2}\mathrm{Tr}\left(u_1\bar{u}_1 v_2\bar{v}_2\right) = g^2\mathrm{Tr}\left[\left(\not{\mathbf{p}}_1 + m\right)\left(\not{\mathbf{p}}_2 - m\right)\right] \\
&= 4g^2\left(\mathbf{p}_1\cdot\mathbf{p}_2 - m^2\right), \quad \text{(scalar coupling)} \\
\sum_{s_1,s_2}\left|i\mathcal{M}\right|^2 &= -g^2\sum_{s_1,s_2}\mathrm{Tr}\left(\bar{u}_1\gamma_5 v_2\bar{v}_2\gamma_5 u_1\right) = -g^2\sum_{s_1,s_2}\mathrm{Tr}\left(u_1\bar{u}_1\gamma_5 v_2\bar{v}_2\gamma_5\right) \\
&= -g^2\mathrm{Tr}\left[\left(\not{\mathbf{p}}_1 + m\right)\gamma_5\left(\not{\mathbf{p}}_2 - m\right)\gamma_5\right] = g^2\mathrm{Tr}\left[\left(\not{\mathbf{p}}_1 + m\right)\left(\not{\mathbf{p}}_2 + m\right)\gamma_5^2\right] \\
&= 4g^2\left(\mathbf{p}_1\cdot\mathbf{p}_2 + m^2\right), \quad \text{(pseudo-scalar coupling)}
\end{aligned}$$

The dot product will be given by $\mathbf{p}_1\cdot\mathbf{p}_2 = E^2 + p^2 = 2E^2 - m^2$. We therefore have

$$\begin{aligned}
\Gamma\left(\phi\to\bar{\psi}\psi\right) &= \frac{1}{2M}\int\frac{p\,d\Omega}{16\pi^2 E_{\text{tot}}}\sum_{s_1,s_2}\left|i\mathcal{M}\right|^2 = \frac{\sqrt{E^2 - m^2}}{32\pi^2 M^2}4g^2\left(2E^2 - 2m^2\right)\int d\Omega \\
&= \frac{g^2\left(\frac{1}{4}M^2 - m^2\right)^{3/2}}{\pi M^2} = \frac{g^2\left(M^2 - 4m^2\right)^{3/2}}{8\pi M^2} \quad \text{(scalar)}, \\
\Gamma\left(\phi\to\bar{\psi}\psi\right) &= \frac{1}{2M}\int\frac{p\,d\Omega}{16\pi^2 E_{\text{tot}}}\sum_{s_1,s_2}\left|i\mathcal{M}\right|^2 = \frac{\sqrt{E^2 - m^2}}{32\pi^2 M^2}4g^2\left(2E^2 - m^2 + m^2\right)\int d\Omega \\
&= \frac{g^2\sqrt{\frac{1}{4}M^2 - m^2}}{4\pi} = \frac{g^2\sqrt{M^2 - 4m^2}}{8\pi} \quad \text{(pseudo-scalar)},
\end{aligned}$$

2. In the $\bar{\psi}\psi\phi$ theory with pseudoscalar coupling, consider the scattering of $\psi(\mathbf{p}_1)\bar{\psi}(\mathbf{p}_2)\to\psi(\mathbf{p}_3)\bar{\psi}(\mathbf{p}_4)$.

(a) Draw the two relevant Feynman diagrams and write the relevant Feynman amplitude.



The relevant Feynman diagrams have been sketched above. It is not hard to see that if you swap the two outgoing arrows, one diagram turns into the other. So there is a relative minus sign between the two diagrams, and the Feynman amplitude is

$$i\mathcal{M}=ig^2\left[\frac{(\bar{u}_3\gamma_5 u_1)(\bar{v}_2\gamma_5 v_4)}{(\mathbf{p}_1-\mathbf{p}_3)^2-M^2}-\frac{(\bar{u}_3\gamma_5 v_4)(\bar{v}_2\gamma_5 u_1)}{(\mathbf{p}_1+\mathbf{p}_2)^2-M^2}\right]$$

(b) Sum and average the Feynman amplitude squared over outgoing and incoming spin states respectively. Simplify the result in the ultrarelativistic limit, so that we can neglect both *m* and *M*. The result should be quite simple.

We want to sum over final spins and average over initial ones. We have

$$\tfrac{1}{4}\sum|i\mathcal{M}|^2$$

$$=\frac{g^4}{4}\sum\left[\frac{(\bar{u}_3\gamma_5 u_1)(\bar{v}_2\gamma_5 v_4)}{(\mathbf{p}_1-\mathbf{p}_3)^2-M^2}-\frac{(\bar{u}_3\gamma_5 v_4)(\bar{v}_2\gamma_5 u_1)}{(\mathbf{p}_1+\mathbf{p}_2)^2-M^2}\right]\left[\frac{(\bar{u}_1\gamma_5 u_3)(\bar{v}_4\gamma_5 v_2)}{(\mathbf{p}_1-\mathbf{p}_3)^2-M^2}-\frac{(\bar{v}_4\gamma_5 u_3)(\bar{u}_1\gamma_5 v_2)}{(\mathbf{p}_1+\mathbf{p}_2)^2-M^2}\right]$$

$$=\frac{g^4}{4}\sum\left\{\begin{array}{l}\dfrac{\mathrm{Tr}(u_3\bar{u}_3\gamma_5 u_1\bar{u}_1\gamma_5)\mathrm{Tr}(v_2\bar{v}_2\gamma_5 v_4\bar{v}_4\gamma_5)}{(2m^2-2\mathbf{p}_1\cdot\mathbf{p}_3-M^2)^2}+\dfrac{\mathrm{Tr}(u_3\bar{u}_3\gamma_5 v_4\bar{v}_4\gamma_5)\mathrm{Tr}(v_2\bar{v}_2\gamma_5 u_1\bar{u}_1\gamma_5)}{(2m^2+2\mathbf{p}_1\cdot\mathbf{p}_2-M^2)^2}\\ -\dfrac{\mathrm{Tr}(u_3\bar{u}_3\gamma_5 u_1\bar{u}_1\gamma_5 v_2\bar{v}_2\gamma_5 v_4\bar{v}_4\gamma_5)+\mathrm{Tr}(u_3\bar{u}_3\gamma_5 v_4\bar{v}_4\gamma_5 v_2\bar{v}_2\gamma_5 u_1\bar{u}_1\gamma_5)}{(2m^2-2\mathbf{p}_1\cdot\mathbf{p}_3-M^2)(2m^2+2\mathbf{p}_1\cdot\mathbf{p}_2-M^2)}\end{array}\right\}$$

$$=\frac{g^4}{4}\left\{\begin{array}{l}\dfrac{\mathrm{Tr}\left[(\not{\mathbf{p}}_3+m)\gamma_5(\not{\mathbf{p}}_1+m)\gamma_5\right]\mathrm{Tr}\left[(\not{\mathbf{p}}_2-m)\gamma_5(\not{\mathbf{p}}_4-m)\gamma_5\right]}{(2m^2-2\mathbf{p}_1\cdot\mathbf{p}_3-M^2)^2}\\ +\dfrac{\mathrm{Tr}\left[(\not{\mathbf{p}}_3+m)\gamma_5(\not{\mathbf{p}}_4-m)\gamma_5\right]\mathrm{Tr}\left[(\not{\mathbf{p}}_2-m)\gamma_5(\not{\mathbf{p}}_1+m)\gamma_5\right]}{(2m^2+2\mathbf{p}_1\cdot\mathbf{p}_2-M^2)^2}\\ -\dfrac{\mathrm{Tr}\left[(\not{\mathbf{p}}_3+m)\gamma_5(\not{\mathbf{p}}_1+m)\gamma_5(\not{\mathbf{p}}_2-m)\gamma_5(\not{\mathbf{p}}_4-m)\gamma_5\right]}{(2m^2-2\mathbf{p}_1\cdot\mathbf{p}_3-M^2)(2m^2+2\mathbf{p}_1\cdot\mathbf{p}_2-M^2)}\\ -\dfrac{\mathrm{Tr}\left[(\not{\mathbf{p}}_3+m)\gamma_5(\not{\mathbf{p}}_4-m)\gamma_5(\not{\mathbf{p}}_2-m)\gamma_5(\not{\mathbf{p}}_1+m)\gamma_5\right]}{(2m^2-2\mathbf{p}_1\cdot\mathbf{p}_3-M^2)(2m^2+2\mathbf{p}_1\cdot\mathbf{p}_2-M^2)}\end{array}\right\}$$

$$\tfrac{1}{4}\sum|i\mathcal{M}|^2 = \frac{g^4}{4}\left\{\begin{array}{l} \dfrac{\operatorname{Tr}\left[(\not{\mathbf{p}}_3+m)(\not{\mathbf{p}}_1-m)\right]\operatorname{Tr}\left[(\not{\mathbf{p}}_2-m)(\not{\mathbf{p}}_4+m)\right]}{\left(2m^2-2\mathbf{p}_1\cdot\mathbf{p}_3-M^2\right)^2} \\ +\dfrac{\operatorname{Tr}\left[(\not{\mathbf{p}}_3+m)(\not{\mathbf{p}}_4+m)\right]\operatorname{Tr}\left[(\not{\mathbf{p}}_2-m)(\not{\mathbf{p}}_1-m)\right]}{\left(2m^2+2\mathbf{p}_1\cdot\mathbf{p}_2-M^2\right)^2} \\ -\dfrac{\operatorname{Tr}\left[(\not{\mathbf{p}}_3+m)(\not{\mathbf{p}}_1-m)(\not{\mathbf{p}}_2-m)(\not{\mathbf{p}}_4+m)\right]}{\left(2m^2-2\mathbf{p}_1\cdot\mathbf{p}_3-M^2\right)\left(2m^2+2\mathbf{p}_1\cdot\mathbf{p}_2-M^2\right)} \\ -\dfrac{\operatorname{Tr}\left[(\not{\mathbf{p}}_3+m)(\not{\mathbf{p}}_4+m)(\not{\mathbf{p}}_2-m)(\not{\mathbf{p}}_1-m)\right]}{\left(2m^2-2\mathbf{p}_1\cdot\mathbf{p}_3-M^2\right)\left(2m^2+2\mathbf{p}_1\cdot\mathbf{p}_2-M^2\right)} \end{array}\right\}$$

We now set all the masses to zero, which simplifies things immensely:

$$\begin{aligned} \tfrac{1}{4}\sum|i\mathcal{M}|^2 &= \frac{g^4}{16}\left\{\begin{array}{l} \dfrac{\operatorname{Tr}(\not{\mathbf{p}}_3\not{\mathbf{p}}_1)\operatorname{Tr}(\not{\mathbf{p}}_2\not{\mathbf{p}}_4)}{(\mathbf{p}_1\cdot\mathbf{p}_3)^2}+\dfrac{\operatorname{Tr}(\not{\mathbf{p}}_3\not{\mathbf{p}}_4)\operatorname{Tr}(\not{\mathbf{p}}_2\not{\mathbf{p}}_1)}{(\mathbf{p}_1\cdot\mathbf{p}_2)^2} \\ +\dfrac{\operatorname{Tr}(\not{\mathbf{p}}_3\not{\mathbf{p}}_1\not{\mathbf{p}}_2\not{\mathbf{p}}_4)+\operatorname{Tr}(\not{\mathbf{p}}_3\not{\mathbf{p}}_4\not{\mathbf{p}}_2\not{\mathbf{p}}_1)}{(\mathbf{p}_1\cdot\mathbf{p}_3)(\mathbf{p}_1\cdot\mathbf{p}_2)} \end{array}\right\} \\ &= \frac{g^4}{4}\left\{\begin{array}{l} \dfrac{4(\mathbf{p}_1\cdot\mathbf{p}_3)(\mathbf{p}_2\cdot\mathbf{p}_4)}{(\mathbf{p}_1\cdot\mathbf{p}_3)^2}+\dfrac{4(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{p}_3\cdot\mathbf{p}_4)}{(\mathbf{p}_1\cdot\mathbf{p}_2)^2} \\ +\dfrac{2\left[(\mathbf{p}_1\cdot\mathbf{p}_3)(\mathbf{p}_2\cdot\mathbf{p}_4)+(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{p}_3\cdot\mathbf{p}_4)-(\mathbf{p}_1\cdot\mathbf{p}_4)(\mathbf{p}_3\cdot\mathbf{p}_2)\right]}{(\mathbf{p}_1\cdot\mathbf{p}_3)(\mathbf{p}_1\cdot\mathbf{p}_2)} \end{array}\right\}. \end{aligned}$$

If we work in the center of mass frame and let the angle of the final particles be θ compared to the initial angles, it isn't hard to work out all the dot products:

$$\begin{aligned} \mathbf{p}_1\cdot\mathbf{p}_2 &= \mathbf{p}_3\cdot\mathbf{p}_4 = 2E^2, \\ \mathbf{p}_1\cdot\mathbf{p}_3 &= \mathbf{p}_2\cdot\mathbf{p}_4 = E^2(1-\cos\theta), \\ \mathbf{p}_1\cdot\mathbf{p}_4 &= \mathbf{p}_2\cdot\mathbf{p}_3 = E^2(1+\cos\theta). \end{aligned}$$

Substituting these in, we have

$$\begin{aligned} \tfrac{1}{4}\sum|i\mathcal{M}|^2 &= \frac{g^4}{4}\left\{\frac{4E^4(1-\cos\theta)^2}{E^4(1-\cos\theta)^2}+\frac{4\cdot 4E^4}{4E^4}+\frac{2\left[E^4(1-\cos\theta)^2+4E^4-E^4(1+\cos\theta)^2\right]}{2E^2E^2(1-\cos\theta)}\right\} \\ &= \frac{g^4}{4}\left\{8+\frac{4-4\cos\theta}{1-\cos\theta}\right\} = \tfrac{1}{4}g^4\{8+4\} = 3g^4 \end{aligned}$$

It can be shown that to reach this equation, we need only set $M = 0$; the assumption $m = 0$ is not necessary.

(c) Calculate the differential and total cross-section.

We have

$$\sigma\left|\Delta\vec{v}\right| = \frac{1}{4E^2}\int\frac{p\,d\Omega}{16\pi^2(2E)}\tfrac{1}{4}\sum\left|i\mathcal{M}\right|^2 = \frac{9g^4E}{128\pi^2E^3}\int d\Omega,$$

$$\frac{d\sigma}{d\Omega} = \frac{9g^4}{256\pi^2E^2},$$

$$\sigma = \frac{9g^4}{64\pi^2E^2}.$$

3. In the $\bar{\psi}\psi\phi$ theory with pseudoscalar coupling, consider the annihilation process $\psi(\mathbf{p})\bar{\psi}(\mathbf{p}')\to\phi(\mathbf{k})\phi(\mathbf{k}')$

(a) Write the two Feynman amplitudes. I found it useful to always write the intermediate propagator in terms of the k **and avoid** k'**. I also found it useful to combine the two terms, as much as possible, before proceeding.**

The two diagrams are sketched above. The diagrams differ by switching external boson lines, so there is a relative plus sign between them. We have

$$\begin{aligned}
i\mathcal{M} &= ig^2\left[\frac{\bar{v}'\gamma_5\left(\not{\mathbf{p}}-\not{\mathbf{k}}+m\right)\gamma_5 u}{(\mathbf{p}-\mathbf{k})^2-m^2}+\frac{\bar{v}'\gamma_5\left(\not{\mathbf{k}}-\not{\mathbf{p}}'+m\right)\gamma_5 u}{(\mathbf{k}-\mathbf{p}')^2-m^2}\right] \\
&= ig^2\left[\frac{\bar{v}'\left(-\not{\mathbf{p}}+\not{\mathbf{k}}+m\right)u}{m^2-2\mathbf{p}\cdot\mathbf{k}+M^2-m^2}+\frac{\bar{v}'\left(\not{\mathbf{p}}'-\not{\mathbf{k}}+m\right)u}{m^2-2\mathbf{p}'\cdot\mathbf{k}+M^2-m^2}\right] \\
&= ig^2\left[\frac{\bar{v}'\not{\mathbf{k}}u}{-2\mathbf{p}\cdot\mathbf{k}+M^2}+\frac{-\bar{v}'\not{\mathbf{k}}u}{-2\mathbf{p}'\cdot\mathbf{k}+M^2}\right] = ig^2\left(\bar{v}'\not{\mathbf{k}}u\right)\left(\frac{1}{M^2-2\mathbf{p}'\cdot\mathbf{k}}-\frac{1}{M^2-2\mathbf{p}\cdot\mathbf{k}}\right)
\end{aligned}$$

where we used the facts that $\not{\mathbf{p}}u = mu$ and $\bar{v}'\not{\mathbf{p}}' = -m\bar{v}'$.

(b) Square and average over incoming spins.

We have

$$\begin{aligned}
\tfrac{1}{4}\sum\left|i\mathcal{M}\right|^2 &= \frac{g^4}{4}\left(\frac{1}{M^2-2\mathbf{p}'\cdot\mathbf{k}}-\frac{1}{M^2-2\mathbf{p}\cdot\mathbf{k}}\right)^2\sum\left(\bar{v}'\not{\mathbf{k}}u\right)\left(\bar{u}\not{\mathbf{k}}v'\right) \\
&= \frac{g^4}{4}\left(\frac{1}{M^2-2\mathbf{p}'\cdot\mathbf{k}}-\frac{1}{M^2-2\mathbf{p}\cdot\mathbf{k}}\right)^2\mathrm{Tr}\left[\left(\not{\mathbf{p}}'-m\right)\not{\mathbf{k}}\left(\not{\mathbf{p}}+m\right)\not{\mathbf{k}}\right] \\
&= g^4\left(\frac{1}{M^2-2\mathbf{p}'\cdot\mathbf{k}}-\frac{1}{M^2-2\mathbf{p}\cdot\mathbf{k}}\right)^2\left[2(\mathbf{p}'\cdot\mathbf{k})(\mathbf{p}\cdot\mathbf{k})-(\mathbf{p}'\cdot\mathbf{p})\mathbf{k}^2-m^2\mathbf{k}^2\right].
\end{aligned}$$

We then substitute $\mathbf{k}^2 = M^2$.

(c) Calculate the differential and total cross-section for this annihilation process. To simplify, ignore the mass *M* (treat the pseudoscalar as massless) but not the mass *m*.

The equations get a lot simpler in the limit, since we then have

$$\tfrac{1}{4}\sum|i\mathcal{M}|^2 = g^4\left(\frac{1}{2\mathbf{p}\cdot\mathbf{k}} - \frac{1}{2\mathbf{p}'\cdot\mathbf{k}}\right)^2\left[2(\mathbf{p}'\cdot\mathbf{k})(\mathbf{p}\cdot\mathbf{k}) - 0 - 0\right] = \frac{g^4}{2}\left(\frac{\mathbf{p}'\cdot\mathbf{k}}{\mathbf{p}\cdot\mathbf{k}} + \frac{\mathbf{p}\cdot\mathbf{k}}{\mathbf{p}'\cdot\mathbf{k}} - 2\right)$$

If the initial momenta are $\mathbf{p} = (E, 0, 0, p)$ and $\mathbf{p}' = (E, 0, 0, -p)$, and we let the angle of the final photon be θ, then $\mathbf{p}\cdot\mathbf{k} = E(E - p\cos\theta)$ and $\mathbf{p}'\cdot\mathbf{k} = E(E + p\cos\theta)$, so

$$\tfrac{1}{4}\sum|i\mathcal{M}|^2 = \frac{g^4}{2}\left(\frac{E - p\cos\theta}{E + p\cos\theta} + \frac{E + p\cos\theta}{E - p\cos\theta} - 2\right) = \frac{2g^4 p^2\cos^2\theta}{E^2 - p^2\cos^2\theta}.$$

This gives us an annihilation differential cross-section of

$$\frac{d\sigma}{d\Omega}|\Delta\vec{v}| = \frac{1}{4E^2}\frac{E}{16\pi^2(2E)}\tfrac{1}{4}\sum|i\mathcal{M}|^2 = \frac{g^4p^2\cos^2\theta}{64\pi^2E^2\left(E^2 - p^2\cos^2\theta\right)}$$

The relative velocity is $2p/E$, so the differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{g^4 p\cos^2\theta}{128\pi^2 E\left(E^2 - p^2\cos^2\theta\right)}$$

There is a subtlety involved with the total cross-section. The final state particles are identical, and therefore when we add up all final states, we must avoid double counting. This is most easily handled by only considering angles between 0 and $\frac{1}{2}\pi$. Letting Maple do the work of integrating for us, we find

$$\sigma = 2\pi\int_0^1 \frac{d\sigma}{d\Omega}d\cos\theta = \frac{g^4 p}{64\pi E}\left\{\frac{E\tanh^{-1}(p/E)}{p^3} - \frac{1}{p^2}\right\} = \frac{g^4}{64\pi}\left\{\frac{1}{p^2}\tanh^{-1}\left(\frac{p}{E}\right) - \frac{1}{pE}\right\}.$$

This can be shown to vanish in the non-relativistic approximation. In the relativistic approximation, it diverges, but only logarithmically.