Quantum Mechanics 741 - Final Equations

The following new equations you should memorize, and understand how to use them:

Angular momentum

$$[J_x, J_y] = i\hbar J_z, \quad \text{etc.}$$

$$\mathbf{J}^2 |j,m\rangle = \hbar^2 (j^2 + j) |j,m\rangle$$

$$J_z |j,m\rangle = \hbar m |j,m\rangle$$

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

$$m = j, j-1, j-2, \ldots, -j$$

Examples of Angular Momentum-Like Operators

$$\mathbf{L}, \mathbf{S}, \mathbf{J} = \mathbf{L} + \mathbf{S}$$

Clebsch-Gordan: $\langle j_1, j_2; m_1, m_2 | j, m \rangle$ non-zero only if:

$$|j_1 - j_2| \le j \le j_1 + j_2$$

$$m_1 + m_2 = m, \quad |m| \le j$$

$$|m_1| \le j_1, \quad |m_2| \le j_2$$

Wave Function with Spin

$$\psi(\mathbf{r},t) = \begin{pmatrix} \psi_s(\mathbf{r}) \\ \psi_{s-1}(\mathbf{r}) \\ \vdots \\ \psi_{-s}(\mathbf{r}) \end{pmatrix}$$

Spin Commutes

$$[S_i, R_j] = 0$$

$$[S_i, P_j] = 0$$

Adding Angular Momentum

$$j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$$

Vector Operators

$$[J_x, V_y] = i\hbar V_z$$

$$[J_x, V_z] = -i\hbar V_y$$

$$[J_x, V_x] = 0 \quad \text{etc.}$$

Spin of proton, neutron, electron: ½

Bosons and Fermions

Bosons: $P|\psi\rangle = |\psi\rangle$

Fermi: $P|\psi\rangle = \eta_P |\psi\rangle$

State Operator

$$\rho \equiv \sum_i f_i |\psi_i\rangle\langle\psi_i|$$

$$\operatorname{Tr}(\rho) = 1, \quad \rho^\dagger = \rho$$

Eigenvalues: $\rho_i \ge 0$

$$\langle A \rangle = \operatorname{Tr}(\rho A)$$

Time Evolution:

$$|\Psi(t)\rangle = U(t,t_0)|\Psi(t_0)\rangle$$

$$i\hbar \frac{\partial}{\partial t} U(t,t_0) = HU(t,t_0)$$

Kinematic Momentum

$$\boldsymbol{\pi} = \mathbf{P} + e\mathbf{A}$$

Trace:

$$\operatorname{Tr}(A) \equiv \sum_i \langle \phi_i | A | \phi_i \rangle$$

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$$

Ground State for N non-interaction particles

Bosons: $E = NE_1$

Fermions: $E = \sum_{i=1}^{N} E_i$

Other things you should know:

- Understanding the concept of spin
- Addition of angular momentum, and when you can use it
- Especially, the resulting angular momentum when you add two angular momenta
- Clebsch-Gordan coefficients:
  - When they don't vanish
  - How you can use them to explicitly add angular momenta
  - How you can use them in the Wigner-Eckart theorem
  - How you can use them to do integrals of three Spherical Harmonics
- How a vector or scalar commutes with **J**
- (in principle) how a spherical tensor commutes with **J**
- Gauge transformations do not change the physics, and therefore theories must be gauge invariant
- (qualitatively) why certain gauge choices are better than others
- Be able to describe the Aharanov-Bohm experiment
- Tensor product spaces
- Generically, what a wave function for multiple particles looks like
- How to find exact eigenstates for *N* interchangeable non-interacting particles
- How to make these eigenstates completely symmetric/anti-symmetric for bosons/fermions
- The spin-statistics theorem

- How the degeneracy pressure calculation is done for fermions
- Generally, how different electrons are filled into orbitals for atoms
- The time evolution operator is linear and unitary; this allows you to prove things from it
- How to use the propagator to get the wave function at time t given it at time t_0
- How to get the state operator as a matrix, or to write it in terms of basis vectors
- How to tell if a state operator is legal; how to tell if it is a pure or mixed state
- How to evolve the state operator and use it for evaluating expectation values
- In the Heisenberg formalism, state vectors don't change, but operators do

The following new equations you need not memorize, but you should know how to use them if given to you:

Rotating with Spin

$$R(\mathcal{R})\psi(\mathbf{r}) = D(\mathcal{R})\psi(\mathcal{R}^T\mathbf{r})$$
$$D(\mathcal{R}(\hat{\mathbf{n}},\theta)) = \exp(-i\theta\hat{\mathbf{n}}\cdot\mathbf{S}/\hbar)$$
$$R(\mathcal{R}(\hat{\mathbf{n}},\theta)) = \exp(-i\theta\hat{\mathbf{n}}\cdot\mathbf{J}/\hbar)$$

Gauge Transformations

$$\mathbf{A}' = \mathbf{A} + \nabla\chi$$
$$U' = U - \partial\chi/\partial t$$
$$\Psi' = \Psi e^{-ie\chi/\hbar}$$

Ehrenfest for E&M:

$$\frac{d}{dt}\langle\mathbf{R}\rangle = \frac{1}{m}\langle\boldsymbol{\pi}\rangle$$
$$\frac{d}{dt}\langle\boldsymbol{\pi}\rangle = \frac{-e}{2m}\langle\boldsymbol{\pi}\times\mathbf{B} - \mathbf{B}\times\boldsymbol{\pi}\rangle - e\langle\mathbf{E}\rangle$$

EM Hamiltonian

$$H = \frac{1}{2m}\boldsymbol{\pi}^2 - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S}$$

EM Fields

$$\mathbf{B} = \nabla\times\mathbf{A}$$
$$\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla U$$

Fermi Energy and Pressure:

$$E_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}$$
$$E = \tfrac{3}{5}NE_F$$
$$P_F = \frac{\hbar^2}{5m}\left(3\pi^2\right)^{2/3} n^{5/3}$$

Wigner-Eckart Theorem

$$\langle\alpha, j, m|T_q^{(k)}|\alpha', j', m'\rangle = \frac{1}{\sqrt{2j+1}}\langle\alpha, j\|T^{(k)}\|\alpha', j'\rangle\langle j', k; m', q| j, m\rangle$$

Integral of Spherical Harmonics: Non-zero requires $l_1 + l_2 - l$ even

$$\int d\Omega\, Y_l^m(\theta,\phi)^* Y_{l_1}^{m_1}(\theta,\phi) Y_{l_2}^{m_2}(\theta,\phi) = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}\langle l_1, l_2; 0,0|l,0\rangle\langle l_1, l_2; m_1, m_2|l,m\rangle$$

Propagator:

$$\Psi(\mathbf{r},t) = \int d^3\mathbf{r}_0 K(\mathbf{r},t;\mathbf{r}_0,t_0)\Psi(\mathbf{r}_0,t_0)$$
$$K(\mathbf{r},t;\mathbf{r}_0,t_0) = \sum_n \phi_n(\mathbf{r}) e^{-iE_n(t-t_0)/\hbar}\phi_n^*(\mathbf{r}_0)$$

Free Propagator (1D):

$$K(x,t;x_0,t_0) = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}}\exp\left[\frac{im(x-x_0)^2}{2\hbar(t-t_0)}\right]$$

State Operators:

$$i\hbar\frac{d}{dt}\rho = [H,\rho]$$

Heisenberg Picture

$$\frac{d}{dt}A(t) = \frac{i}{\hbar}[H(t), A(t)]$$