Quantum Mechanics 742 - Final Equations

The following new equations you should memorize, and understand how to use them:

Fermi's Golden Rule: $\mathcal{T}_{FI} = W_{FI} + \cdots$

$$\Gamma(I \to F) = 2\pi\hbar^{-1} \left|\mathcal{T}_{FI}\right|^2 \delta(E_F - E_I)$$

Sudden: $T\Delta E \ll \hbar$

$$P(I \to F) = \left|\langle \psi_F | \psi_I \rangle\right|^2$$

Adiabatic: $T\Delta E \gg \hbar$

$$P(I \to F) = \delta_{IF}$$

Time Dependent Perturbation Theory

$$H = H_0 + W(t)$$

$$H_0 |\phi_n\rangle = E_n |\phi_n\rangle$$

$$\omega_{nm} \equiv (E_n - E_m)/\hbar$$

$$W_{nm}(t) \equiv \langle \phi_n | W(t) | \phi_m \rangle$$

$$P(I \to F) = \left|S_{FI}\right|^2$$

Electric Dipole Moment

$$\mathbf{r}_{nm} \equiv \langle \phi_n | \mathbf{R} | \phi_m \rangle$$

EM waves

$$\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \cdot \boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma'} = \delta_{\sigma\sigma'}$$

$$\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \cdot \mathbf{k} = 0$$

$$\omega_k = ck$$

Photon Creation and Annihilation Operators

$$\left[a_{\mathbf{k}\sigma}, a^\dagger_{\mathbf{k}'\sigma'}\right] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'}$$

$$a_{\mathbf{k}\sigma} |n, \mathbf{k}, \sigma\rangle = \sqrt{n}\, |n-1, \mathbf{k}, \sigma\rangle$$

$$a^\dagger_{\mathbf{k}\sigma} |n, \mathbf{k}, \sigma\rangle = \sqrt{n+1}\, |n+1, \mathbf{k}, \sigma\rangle$$

WKB approximation: $n = 0,1,2,\ldots$

$$\int_a^b \sqrt{2m\left[E - V(x)\right]}\, dx = \left(n + \tfrac{1}{2}\right)\pi\hbar$$

EM Hamiltonian

$$H = \sum_{\mathbf{k},\sigma} \hbar\omega_k a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}$$

Coulomb Gauge $\nabla \cdot \mathbf{A} = 0$

Converting Finite $\to$ Infinite

$$\lim_{V\to\infty}\left[\frac{1}{V}\sum_{\mathbf{k}} f(\mathbf{k})\right] = \int f(\mathbf{k}) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Other things you should know:

- Converting from Cartesian to spherical coordinates and back again
- Doing integrals in spherical coordinates
- How to do integrals like $\int f(x)\delta\left[g(x)\right]dx$
- In the adiabatic approximation, make sure the eigenstates you use are correct eigenstates of the initial and final Hamiltonian.
- In the adiabatic approximation: The lowest energy state goes to the lowest state, second lowest to second lowest, etc.
    - However, if there is a symmetry that always commutes with the Hamiltonian, then first sort states by eigenvalues of that symmetry
- For harmonic perturbations, make sure you know how to extract W given $W(t)$, and don't get the two confused
    - In contrast, when the perturbation is independent of time, $W(t) = W$
- How to compute expressions like $\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \cdot \mathbf{r}_{FI}$ for each of the two possible polarizations
- How to average (or sum) over polarizations, and average (or integrate) over angles
- How to take the limit $V \to \infty$, and turn sums over final states into integrals you can do
- What a quantum state like $|n_1, \mathbf{k}_1, \sigma_1; n_2, \mathbf{k}_2, \sigma_2\rangle$ means, or $|\phi_a; n_1, \mathbf{k}_1, \sigma_1; n_2, \mathbf{k}_2, \sigma_2\rangle$ means
- Understanding that after quantizing the EM field, electric and magnetic fields are now operators that are functions of **r**, but not t
- How to write out sums with EM fields and expectation values where the sums collapse to few terms
- How to find the energy of a system of photons, or photons plus an atom
    - Related: the time dependence of such a system
- How to create or annihilate one photon from any state with any number of photons
- Qualitatively, what our diagrams mean in our computations
- Why we concentrate only on certain diagrams when scattering near a resonance

The following new equations you need not memorize, but you should know how to use them if given to you:

Time-dependent Perturbation Theory

$$S_{FI} = \delta_{FI} + (i\hbar)^{-1}\int_0^T dt\, W_{FI}(t) e^{i\omega_{FI}t} + (i\hbar)^{-2}\sum\int_0^T dt\, W_{Fn}(t) e^{i\omega_{Fn}t}\int_0^t dt'\, W_{nI}(t') e^{i\omega_{nI}t'} + \cdots$$

Electric Dipole Absorption

$$\Gamma_{E1} = 4\pi^2\alpha\hbar^{-1}\mathcal{I}(\omega_{FI})\left|\boldsymbol{\varepsilon}\cdot\mathbf{r}_{FI}\right|^2$$

Harmonic Perturbations:

$$W(t) = We^{-i\omega t} + W^\dagger e^{i\omega t}, \quad W_{FI} \equiv \langle\phi_F|W|\phi_I\rangle$$

$$\Gamma(I\to F) = \begin{cases} 2\pi\hbar^{-1}\left|W_{FI}\right|^2\delta(E_F - E_I - \hbar\omega) & \text{if } E_F > E_I \\ 2\pi\hbar^{-1}\left|W_{FI}^\dagger\right|^2\delta(E_F - E_I + \hbar\omega) & \text{if } E_F < E_I \end{cases}$$

Scattering Away from Resonance

$$\mathcal{T}_{FI} = \frac{-e^2}{\varepsilon_0 V}\sum_n \frac{\omega\omega_{nI}}{\omega_{nI}^2 - \omega^2}\left(\boldsymbol{\varepsilon}_F^*\cdot\mathbf{r}_{nI}^*\right)\left(\boldsymbol{\varepsilon}_I\cdot\mathbf{r}_{nI}\right)$$

Transition matrix:

$$\mathcal{T}_{FI} = W_{FI} + \lim_{\varepsilon\to 0^+}\left[\sum_n \frac{W_{Fn}W_{nI}}{(E_I - E_n + i\varepsilon)} + \sum_n\sum_m \frac{W_{Fn}W_{nm}W_{mI}}{(E_I - E_m + i\varepsilon)(E_I - E_n + i\varepsilon)} + \cdots\right]$$

Spontaneous Decay

$$\Gamma = \frac{4\alpha}{3c^2}\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2$$

Scattering Near Resonance

$$\sigma = \frac{8\pi\alpha^2\omega_{nI}^4\left|\mathbf{r}_{nI}\right|^4}{3c^2\left[(\omega-\omega_{nI})^2 + \frac{1}{4}\Gamma^2\right]}$$

Electromagnetic Operators

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} + a_{\mathbf{k}\sigma}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^* e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

$$\mathbf{E}(\mathbf{r}) = \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,i\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\sigma}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^* e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

$$\mathbf{B}(\mathbf{r}) = \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\,i\mathbf{k}\times\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k}\sigma}^\dagger\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^* e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$