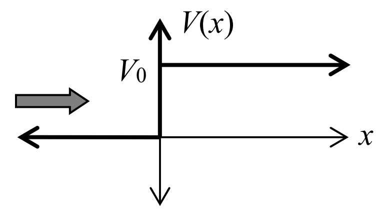
# Quantum Mechanics
# Graduate Exam

Summer, 2020

Each problem is worth 25 points. The points for individual parts are marked in square brackets. **To ensure full credit, show your work.** Do any four (4) of the following five (5) problems. If you attempt all 5 problems you must clearly state which 4 problems you want to have graded.

1. A particle of mass $m$ with energy $E$ in one dimension impacts from the left on a potential $V(x) = \lambda\delta(x) + V_0\theta(x)$, where $E < V_0$, $\delta(x)$ is the Dirac-delta function and $\theta(x)$ is the Heaviside function $\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$

   This potential is sketched at right.




   (a) [9] Find the general solution of Schrödinger's equation for each region $x < 0$ and $x > 0$. Which pieces represent the incoming and reflected wave? Which pieces, if any, must vanish?
   (b) [8] By integrating Schrödinger's equation across the boundary, find boundary conditions on the wave function and its derivative at $x = 0$.
   (c) [8] Solve the equations you have found to find the amplitude of the reflected wave compared to the incoming wave.

2. Under certain conditions, the Hamiltonian describing a neutrino is given in a certain basis by

$$H = \hbar\omega \begin{pmatrix} 3 & \frac{1}{2}\pi \\ \frac{1}{2}\pi & 3 \end{pmatrix}.$$

   (a) [10] Suppose at $t = 0$ the system is in the state $|\Psi(t=0)\rangle = |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. What would be the possible energies one could measure, and what would be their corresponding probabilities?
   (b) [5] Find an expression of the state $|\Psi(t)\rangle$ at an arbitrary time.
   (c) [5] At time $\omega t = 5$, what are the possible values that one would obtain if the energy were measured and what is the probability of each measurement?
   (d) [5] At time $\omega t = 5$, suppose instead that we measure which of the two states in this basis it is in. What is the probability it is in the state $|2\rangle$?

3. A particle of mass $m$ in one dimension lies in the potential $V(x)=\begin{cases}\frac{1}{2}m\omega_1^2x^2 & \text{if } x>0,\\ \frac{1}{2}m\omega_2^2x^2 & \text{if } x<0.\end{cases}$

   Estimate the energy of the ground state using the unnormalized trial wave function $\psi = e^{-Ax^2/2}$. Show that it has the exact correct answer if $\omega_1=\omega_2$.

4. The ground state of hydrogen is $\psi(r,\theta,\phi)=Ne^{-r/a}$.
   (a) [5] What is the correct normalization $N$?
   (b) [13] What are the expectation values of $\langle Z\rangle$, $\langle Z^2\rangle$, $\langle P_z\rangle$, and $\langle P_z^2\rangle$?
   (c) [7] Find the uncertainties $\Delta z$ and $\Delta p_z$ and check that they satisfy the uncertainty relation.

5. A particle of mass $m$ in 3D is in a potential $V(x,y,z)=\begin{cases}\frac{1}{2}m\omega^2\left(x^2+y^2\right) & \text{for } |z|<a,\\ \infty & \text{for } |z|>a.\end{cases}$
   (a) [10] Write a formula for the energy of all the eigenstates of this potential. Also, write the wave function explicitly for the ground state. **You do not need to normalize it.**
   (b) [8] It is found that the first excited state is triply degenerate; that is, there are three states with the same energy. From this, deduce a relationship between $m$, $a$, and $\omega$.
   (c) [7] To this potential is added a small additional potential $W=\lambda YZ$. What effect will this have on the energies of the triply degenerate state to first order? In particular, write down the relevant $3\times 3$ perturbation matrix, and determine which components must vanish. You do not need to calculate the non-vanishing components, nor must you do anything else with them.

**1D Harmonic Oscillator:** Ground state: $\psi_0(x)\propto\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$

**Spherical Coordinates:**

$$x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta,\quad \frac{\partial}{\partial z}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta},$$

$$\frac{\partial}{\partial x}=\sin\theta\cos\phi\frac{\partial}{\partial r}-\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi},\quad \frac{\partial}{\partial y}=\sin\theta\sin\phi\frac{\partial}{\partial r}-\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

**Possibly Helpful Integrals**

$$\int d\Omega=4\pi,\quad \int\cos\theta\, d\Omega=0,\quad \int\cos^2\theta\, d\Omega=\frac{4\pi}{3},\quad \int\sin\theta\, d\Omega=\pi^2,\quad \int\sin^2\theta\, d\Omega=\frac{8\pi}{3}.$$

$$\int_0^\infty r^n e^{-r/b}dr=b^{n+1}n!,\quad \int_0^\infty e^{-Bx^2}dx=\frac{1}{2}\sqrt{\frac{\pi}{B}},\quad \int_0^\infty xe^{-Bx^2}dx=\frac{1}{2B},\quad \int_0^\infty x^2e^{-Bx^2}dx=\frac{1}{4}\sqrt{\frac{\pi}{B^3}}.$$