Quantum Mechanics
Graduate Exam

Summer, 2023

Each problem is worth 25 points. The points for individual parts are marked in square brackets. **To ensure full credit, show your work.** Do any four (4) of the following five (5) problems. If you attempt all 5 problems, you must clearly state which 4 problems you want to have graded. Some possibly helpful formulas are given at the end of the exam.

1. Consider a particle of mass m in a two-dimensional infinite square well with allowed region $0 < x < a$ and $0 < y < a$. In addition, there is a small perturbation with potential

$$W(x,y) = \begin{cases} w_0 & \text{if } 0 < x < \tfrac{1}{2}a, 0 < y < \tfrac{1}{2}a, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) [5] What are the unperturbed energy eigenvalues and corresponding wave functions?
   (b) [8] Calculate, to first order in w_0, the perturbed energy of the ground state.
   (c) [12] Same question for the first excited state(s). Give the corresponding wave function(s) to zeroth order in w_0.

2. A particle of mass m lies in an infinite square well with allowed region $0 < x < a$. At $t = 0$, the wave function is given by

$$\Psi(x,t=0) = \begin{cases} \sqrt{2/a} & \text{for } 0 < x < \tfrac{1}{2}a, \\ 0 & \text{for } \tfrac{1}{2}a < x < a. \end{cases}$$

   (a) [3] Will the particle remain localized in the left half of the well at later times?
   (b) [7] The energy of the particle is measured. What is the probability that the result is the ground state energy E_1? What is the probability that the result is the first excited state energy E_2?
   (c) [7] Calculate the probability that the energy yields the n'th state E_n.
   (d) [8] Show that the sum of the probabilities you found in part (c) is 1. The formula below may be useful.

$$\sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

3. An electron of mass m is bound to a proton at the origin due to a potential $V(r) = -B/r$. Estimate the ground-state energy of this potential using the variational principle with trial wave function $\psi(r) = \begin{cases} a - r & \text{if } r < a, \\ 0 & \text{if } r > a. \end{cases}$ and compare to the exact energy $E_g = -\dfrac{mB^2}{2\hbar^2}$.

4. A single particle of mass m is in the 2D anisotropic harmonic oscillator with potential $V(x,y)=\frac{1}{2}m\omega^2x^2+\frac{1}{2}m(2\omega)^2y^2$.
   (a) [5] What are the energies of the states $|n_x,n_y\rangle$?
   (b) [8] At $t = 0$, the system is in the state $|\Psi(t=0)\rangle=\frac{1}{\sqrt{3}}\left(|1,0\rangle+\sqrt{2}|0,1\rangle\right)$. What is $|\Psi(t)\rangle$?
   (c) [12] What is the expectation value of $\langle XY\rangle$ for the state $|\Psi(t)\rangle$ from part (b) at all times t?

5. A particle of mass m in the one-dimensional potential $V(x)=-B\delta(x)$ has only one bound state, with normalized wave function $\psi(x)=\sqrt{\lambda}e^{-\lambda|x|}$, with $\lambda\hbar^2=Bm$. A particle is initially in this state at $t = 0$, but then the potential is shifted to a new position, so now $V(x)=-B\delta(x-a)$. What is the probability it remains bound if the shift is
   (a) [8] gradual, or
   (b) [17] sudden.

1D Harmonic Oscillator
potential $\frac{1}{2}m\omega^2x^2$

$$X=\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right),\quad a|n\rangle=\sqrt{n}|n-1\rangle,\quad a^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle.$$

Operators in Spherical Coordinates

$$\nabla\psi=\hat{\mathbf{r}}\frac{\partial\psi}{\partial r}+\frac{1}{r}\hat{\boldsymbol{\theta}}\frac{\partial\psi}{\partial\theta}+\frac{1}{r\sin\theta}\hat{\phi}\frac{\partial\psi}{\partial\phi}$$

$$\nabla^2\psi=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Possibly Helpful Integrals: In all formulas below, n and p are assumed to be positive integers.

$$\int\sin(\alpha x)\,dx=-\alpha^{-1}\cos(\alpha x),\quad\int\cos(\alpha x)\,dx=\alpha^{-1}\sin(\alpha x),\quad\int e^{\alpha x}dx=\alpha^{-1}e^{\alpha x}.$$

$$\int_0^\infty x^n e^{-Ax^2}dx=\tfrac{1}{2}A^{-(n+1)/2}\Gamma\left(\tfrac{n+1}{2}\right),\quad\int_0^\infty x^n e^{-Ax}dx=A^{-(n+1)}\Gamma(n+1),$$

$$\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi},\quad\Gamma(1)=1,\quad\Gamma\left(\tfrac{3}{2}\right)=\tfrac{1}{2}\sqrt{\pi},\quad\Gamma(2)=1,\quad\Gamma\left(\tfrac{5}{2}\right)=\tfrac{3}{4}\sqrt{\pi},\quad\Gamma(3)=2.$$

$$\int_0^a\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi px}{a}\right)dx=\int_0^a\cos\left(\frac{\pi nx}{a}\right)\cos\left(\frac{\pi px}{a}\right)dx=\frac{a}{2}\delta_{np},$$

$$\int_0^a\sin\left(\frac{\pi nx}{a}\right)\cos\left(\frac{\pi px}{a}\right)dx=\frac{an}{\pi\left(n^2-p^2\right)}\left[1-(-1)^{n+p}\right].$$

$$\int_0^{a/2}\sin^2\left(\frac{\pi nx}{a}\right)dx=\int_0^{a/2}\cos^2\left(\frac{\pi nx}{a}\right)dx=\frac{a}{4},\quad\int_0^{a/2}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx=\frac{2a}{3\pi}.$$