Quantum Mechanics Graduate Exam Summer 2024

Each problem is worth 25 points. The points for individual parts are marked in square brackets. **To ensure full credit, show your work.** Do any four (4) of the following five (5) problems. If you attempt all 5 problems, you must clearly state which 4 problems you want to have graded. Some possibly helpful formulas are given at the end of the exam.

1. An electron is at rest in an oscillating magnetic field $\mathbf{B} = B_0 \cos(\omega t)\hat{\mathbf{z}}$, where B_0 and ω are constants.
   (a) [2] Construct the Hamiltonian matrix for this system.
   (b) [10] The electron starts out at $t = 0$ in the spin-up state with respect to the x-axis (that is, $\left|\Psi(0)\right\rangle = \left|+_x\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+_z\right\rangle + \left|-_z\right\rangle\right)$). Determine $\left|\Psi(t)\right\rangle$ at subsequent times.
   (c) [6] Find the probability of getting $-\frac{1}{2}\hbar$ if you measure S_x at time t.
   (d) [7] What is the minimum field (B_0) required to force a complete flip in S_x?

2. A particle of mass m is in the ground state of an infinite square well of length a, with allowed region $0 \le x \le a$. Suddenly, the walls expand so that the right wall is now at $x = 2a$ (and the left wall stays in the same place).
   (a) [5] What is the probability that, immediately after the wall has expanded, the particle will be found between (i) $0 \le x \le a$, and (ii) $a \le x \le 2a$?
   (b) [15] What is the probability that if you measured the energy you would get $E = \dfrac{\pi^2\hbar^2}{8ma^2}$?
   (c) [5] Assume that you got the energy given in part (b). If the right wall is now slowly returned to $x = a$, what would be the most likely result of an energy measurement, and what would be the probability of obtaining that result?

3. A particle of mass m lies in a three-dimensional attractive power-law potential, $V(r) = -Ar^{-3/2}$. Using the variational principle, estimate the energy of the ground state using the unnormalized trial wave function $\psi(\mathbf{r}) = e^{-\lambda r}$. Can we be sure that this potential actually has a bound state?

4. A particle of mass m lies in a symmetric 2D harmonic oscillator with potential $V = \frac{1}{2}m\omega^2\left(x^2 + y^2\right)$. In addition, there is a small perturbation $W = \gamma L_z = \gamma\left(xp_y - yp_x\right)$.
   (a) [4] Find the eigenstates and energies of the unperturbed Hamiltonian. You do not need to give explicit forms for these eigenstates, it is sufficient to simply label them as, say, $\left|n_x, n_y\right\rangle$. Check that the ground state is non-degenerate, but the first excited state is degenerate.
   (b) [9] Show that the ground state is an exact eigenstate of the perturbed Hamiltonian
   (c) [12] Find the energies of the first excited states to first order in γ, and the corresponding eigenstates to leading order.

5. A measurement corresponding to observable A has two normalized eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ with eigenvalues a_1 and a_2 respectively. A second measurement corresponding to observable B, has two normalized eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$ with eigenvalues b_1 and b_2 respectively. The eigenstates are related by

$$|\psi_1\rangle = \tfrac{3}{5}|\phi_1\rangle + \tfrac{4}{5}|\phi_2\rangle \qquad ; \qquad |\psi_2\rangle = \tfrac{4}{5}|\phi_1\rangle - \tfrac{3}{5}|\phi_2\rangle$$

(a) [8] Observable A is measured, and the value a_1 is obtained. What is the state of the wave function $|\Psi\rangle$ **immediately** after this measurement?

(b) [8] If B is now measured, what are the possible results, and what are their probabilities?

(c) [9] Assume in part (b) that the result yielded result b_1. Right after the measurement of B, A is measured again. What is the probability of getting the value a_1 again? How would your answer be different if the measurement in part (b) had been b_2 instead?

1D Harmonic Oscillator potential $\frac{1}{2}m\omega^2x^2$

$$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right), \quad P = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right), \quad a|n\rangle = \sqrt{n}\,|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

Laplacian in Spherical Coordinates

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Spin and Magnetic Fields

$$H = \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S}, \quad \mathbf{S} = \tfrac{1}{2}\hbar\boldsymbol{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Momentum Eigenstates: $\phi_p(x) = \dfrac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$

Possibly Helpful Integrals: In all formulas below, α and β are distinct and $n > -1$.

$$\int \sin(\alpha x)\,dx = -\alpha^{-1}\cos(\alpha x), \quad \int \cos(\alpha x)\,dx = \alpha^{-1}\sin(\alpha x), \quad \int e^{\alpha x}dx = \alpha^{-1}e^{\alpha x},$$

$$\int \sin(\alpha x)\sin(\beta x)\,dx = \frac{\sin\left[(\alpha-\beta)x\right]}{2(\alpha-\beta)} - \frac{\sin\left[(\alpha+\beta)x\right]}{2(\alpha+\beta)}, \quad \int \sin^2(\alpha x)\,dx = \frac{x}{2} - \frac{\sin(2\alpha x)}{4\alpha},$$

$$\int \cos(\alpha x)\cos(\beta x)\,dx = \frac{\sin\left[(\alpha-\beta)x\right]}{2(\alpha-\beta)} + \frac{\sin\left[(\alpha+\beta)x\right]}{2(\alpha+\beta)}, \quad \int \cos^2(\alpha x)\,dx = \frac{x}{2} + \frac{\sin(2\alpha x)}{4\alpha},$$

$$\int \sin(\alpha x)\cos(\beta x)\,dx = \frac{\cos\left[(\alpha+\beta)x\right]}{2(\alpha+\beta)} + \frac{\cos\left[(\alpha-\beta)x\right]}{2(\alpha-\beta)}, \quad \int \sin(\alpha x)\cos(\alpha x)\,dx = -\frac{\cos(2\alpha x)}{4\alpha}.$$

$$\int_0^\infty x^n e^{-\alpha x^2}dx = \tfrac{1}{2}\alpha^{-(n+1)/2}\Gamma\left(\tfrac{n+1}{2}\right), \quad \int_0^\infty x^n e^{-\alpha x}dx = \alpha^{-(n+1)}\Gamma(n+1),$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(1) = 1, \quad \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma(2) = 1, \quad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}, \quad \Gamma(3) = 2.$$