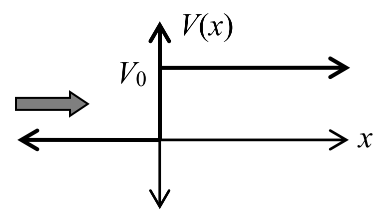
Solutions to Quantum Mechanics Graduate Exam

Summer, 2020

Each problem is worth 25 points. The points for individual parts are marked in square brackets. **To ensure full credit, show your work.** Do any four (4) of the following five (5) problems. If you attempt all 5 problems you must clearly state which 4 problems you want to have graded.

1. A particle of mass *m* with energy E in one dimension impacts from the left on a potential $V(x)=\lambda\delta(x)+V_0\theta(x)$, where $E<V_0$, $\delta(x)$ is the Dirac-delta function and $\theta(x)$ is the Heaviside function $\theta(x)=\begin{cases}1 & \text{if } x>0,\\ 0 & \text{if } x<0.\end{cases}$

This potential is sketched at right.



(a) [9] Find the general solution of Schrödinger's equation for each region *x* < 0 and *x* > 0. Which pieces represent the incoming and reflected wave? Which pieces, if any, must vanish?

Schrödinger's equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)+V(x)\psi(x)=E\psi(x)$$

For $x<0$, the potential vanishes, while for $x>0$ it is the constant V_0, so we have

$$\frac{d^2}{dx^2}\psi(x)=-\frac{2mE}{\hbar^2}\psi(x)\quad \text{for } x<0,$$
$$\frac{d^2}{dx^2}\psi(x)=\frac{2m(V_0-E)}{\hbar^2}\psi(x)\quad \text{for } x>0.$$

If we define $2mE/\hbar^2=k^2$ and $2m(V_0-E)/\hbar^2=\alpha^2$, then the solutions look like $e^{\pm ikx}$ for $x<0$ and $e^{\pm\alpha x}$ for $x>0$. The most general solution is therefore

$$\psi(x)=\begin{cases}Ae^{ikx}+Be^{-ikx} & \text{for } x<0,\\ Ce^{\alpha x}+De^{-\alpha x} & \text{for } x>0.\end{cases}$$

The *A* term represents the incoming wave, and *B* is the reflected wave. The *C* term is unacceptable, as it blows up at infinity, and the *D* term is called the evanescent wave.

(b) [8] By integrating Schrödinger's equation across the boundary, find boundary conditions on the wave function and its derivative at *x* = 0.

Integrating *x* over a tiny region $(-\varepsilon,\varepsilon)$ across the boundary, we have

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{d^2}{dx^2}\psi(x)\,dx+\int_{-\varepsilon}^{\varepsilon}\left[\lambda\delta(x)+V_0\theta(x)\right]\psi(x)\,dx=E\int_{-\varepsilon}^{\varepsilon}\psi(x)\,dx$$

We are thinking of ε as very small, so we ignore any term that is small in the limit $\varepsilon \to 0$. The wave function is finite, so the term on the right will vanish in this limit. The same applies to the Heaviside term in the potential. However, the Dirac δ-function will contribute, and we can do the first term by using the fundamental theorem of calculus. We therefore have

$$-\frac{\hbar^2}{2m}\frac{d\psi}{dx}\bigg|_{-\varepsilon}^{\varepsilon} + \lambda\psi(0) = 0,$$

$$\psi'(+\varepsilon) - \psi'(-\varepsilon) = \frac{2m\lambda}{\hbar^2}\psi(0).$$

Because there is a finite discontinuity in the derivative, the function must be continuous, so our two conditions are

$$\psi'(+\varepsilon) - \psi'(-\varepsilon) = \frac{2m\lambda}{\hbar^2}\psi(0) \quad \text{and} \quad \psi(+\varepsilon) = \psi(-\varepsilon).$$

(c) [8] Solve the equations you have found to find the amplitude of the reflected wave compared to the incoming wave.

We write these two expressions out in terms of our explicit form of the wave function, substituting $\varepsilon \to 0$, and recalling that $C = 0$ to yield

$$-\alpha D - (ikA - ikB) = \frac{2m\lambda}{\hbar^2}D \quad \text{and} \quad A + B = D.$$

Using the equation on the right to eliminate D on the left, we have

$$(ikB - ikA) - \alpha A - \alpha B = \frac{2m\lambda}{\hbar^2}(A + B),$$

$$\left(ik - \alpha - \frac{2m\lambda}{\hbar^2}\right)B = \left(ik + \alpha + \frac{2m\lambda}{\hbar^2}\right)A,$$

$$\frac{B}{A} = \frac{ik + \alpha + 2m\lambda/\hbar^2}{ik - \alpha - 2m\lambda/\hbar^2}.$$

We can substitute $2mE/\hbar^2 = k^2$ and $2m(V_0 - E)/\hbar^2 = \alpha^2$ to rewrite this as

$$\frac{B}{A} = \frac{i\sqrt{E} + \sqrt{V_0 - E} + \sqrt{2m}\lambda/\hbar}{i\sqrt{E} - \sqrt{V_0 - E} - \sqrt{2m}\lambda/\hbar}.$$

2. Under certain conditions, the Hamiltonian describing a neutrino is given in a certain basis by

$$H = \hbar\omega \begin{pmatrix} 3 & \tfrac{1}{2}\pi \\ \tfrac{1}{2}\pi & 3 \end{pmatrix}.$$

(a) [10] Suppose at $t = 0$ the system is in the state $|\Psi(t=0)\rangle = |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. What would be the possible energies one could measure, and what would be their corresponding probabilities?

We first need to find the eigenvalues and eigenvectors of this matrix. Ignoring the factor of $\hbar\omega$, we must solve the characteristic equation

$$0 = \det \begin{pmatrix} 3-\lambda & \tfrac{1}{2}\pi \\ \tfrac{1}{2}\pi & 3-\lambda \end{pmatrix} = (3-\lambda)^2 - \left(\tfrac{1}{2}\pi\right)^2,$$

$$(\lambda - 3)^2 = \left(\tfrac{1}{2}\pi\right)^2,$$

$$\lambda = 3 \pm \tfrac{1}{2}\pi.$$

These eigenvalues must then be multiplied by $\hbar\omega$. The eigenvectors can be found by inserting an arbitrary vector and satisfying the eigenvector equation, so we have

$$\begin{pmatrix} 3 & \tfrac{1}{2}\pi \\ \tfrac{1}{2}\pi & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(3 \pm \tfrac{1}{2}\pi\right) \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

$$3\alpha + \tfrac{1}{2}\pi\beta = 3\alpha \pm \tfrac{1}{2}\pi\alpha \quad \text{and} \quad \tfrac{1}{2}\pi\alpha + 3\beta = 3\beta \pm \tfrac{1}{2}\pi\beta,$$

$$\beta = \pm\alpha \quad \text{and} \quad \alpha = \pm\beta.$$

As always, we get one redundant equation. The normalization condition demands that $1 = |\alpha|^2 + |\beta|^2 = 2|\alpha|^2$, which we can satisfy by choosing $\alpha = \frac{1}{\sqrt{2}}$ and then $\beta = \pm\frac{1}{\sqrt{2}}$. Putting it all together, our eigenvectors and eigenvalues are

$$|\phi_\pm\rangle = \left|\left(3 \pm \tfrac{1}{2}\pi\right)\hbar\omega\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$

The amplitude for it being in each of these states is just $\langle \phi_\pm | \Psi(0) \rangle = \pm\frac{1}{\sqrt{2}}$. The probability of getting the energies is then the magnitude squared of this, so we have

$$P\left(E = \left(3 + \tfrac{1}{2}\pi\right)\hbar\omega\right) = \frac{1}{2} \quad \text{and} \quad P\left(E = \left(3 - \tfrac{1}{2}\pi\right)\hbar\omega\right) = \frac{1}{2}.$$

(b) [5] Find an expression of the state $|\Psi(t)\rangle$ at an arbitrary time.

At $t = 0$, the state can be written as

$$\left|\Psi(0)\right\rangle = \sum_i \left|\phi_i\right\rangle\left\langle\phi_i\middle|\Psi(0)\right\rangle = \left|\phi_+\right\rangle\left\langle\phi_+\middle|\Psi(0)\right\rangle + \left|\phi_-\right\rangle\left\langle\phi_-\middle|\Psi(0)\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|\phi_+\right\rangle - \left|\phi_-\right\rangle\right).$$

Energy eigenstates just pick up a phase $e^{-iE_n t/\hbar}$, so at later times we have

$$\left|\Psi(t)\right\rangle = \tfrac{1}{\sqrt{2}}\left[\left|\phi_+\right\rangle e^{-i\left(3+\frac{1}{2}\pi\right)\omega t} - \left|\phi_-\right\rangle e^{-i\left(3-\frac{1}{2}\pi\right)\omega t}\right].$$

(c) [5] At time $\omega t = 5$, what are the possible values that one would obtain if the energy were measured and what is the probability of each measurement?

The only possible values are the energy eigenvalues, $\hbar\omega\left(3 \pm \tfrac{1}{2}\pi\right)$, and the probabilities are simply $\left|\left\langle\phi_\pm\middle|\Psi(t)\right\rangle\right|^2$. The introduction of a phase has no effect on this, so the probabilities are exactly the same as before,

$$P\left(E = \left(3+\tfrac{1}{2}\pi\right)\hbar\omega\right) = \frac{1}{2} \quad \text{and} \quad P\left(E = \left(3-\tfrac{1}{2}\pi\right)\hbar\omega\right) = \frac{1}{2}.$$

(d) [5] At time $\omega t = 5$, suppose instead that we measure which of the two states in this basis it is in. What is the probability it is in the state $\left|2\right\rangle$?

The probability is given by $\left|\left\langle 2\middle|\Psi(t)\right\rangle\right|^2$, so we have

$$P(2) = \left|\left\langle 2\middle|\Psi(t)\right\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\left\langle 2\middle|\phi_+\right\rangle e^{-i\left(3+\frac{1}{2}\pi\right)\omega t} - \tfrac{1}{\sqrt{2}}\left\langle 2\middle|\phi_-\right\rangle e^{-i\left(3-\frac{1}{2}\pi\right)\omega t}\right|^2 = \left|\tfrac{1}{2}e^{-i\left(3+\frac{1}{2}\pi\right)\omega t} + \tfrac{1}{2}e^{-i\left(3-\frac{1}{2}\pi\right)\omega t}\right|^2$$

$$= \frac{1}{4}\left[e^{i\left(3+\frac{1}{2}\pi\right)\omega t} + e^{i\left(3-\frac{1}{2}\pi\right)\omega t}\right]\left[e^{-i\left(3+\frac{1}{2}\pi\right)\omega t} + e^{-i\left(3-\frac{1}{2}\pi\right)\omega t}\right] = \frac{1}{4}\left[1 + e^{-i\pi\omega t} + e^{i\pi\omega t} + 1\right] = \frac{1}{2} + \frac{1}{2}\cos(\pi\omega t).$$

Evaluating this expression at $\omega t = 5$, $\cos(\pi\omega t) = \cos(5\pi) = -1$, and we have $P(2) = 0$.

3. A particle of mass *m* in one dimension lies in the potential $V(x)=\begin{cases}\frac{1}{2}m\omega_1^2x^2 & \text{if } x>0,\\ \frac{1}{2}m\omega_2^2x^2 & \text{if } x<0.\end{cases}$ **Estimate the energy of the ground state using the unnormalized trial wave function** $\psi=e^{-Ax^2/2}$**. Show that it has the exact correct answer if** $\omega_1=\omega_2$**.**

To use the variational approach, we must calculate the following quantities:

$$\langle\psi|\psi\rangle=\int_{-\infty}^{\infty}|\psi(x)|^2\,dx=\int_{-\infty}^{\infty}e^{-Ax^2}dx=2\int_0^{\infty}e^{-Ax^2}dx=\frac{2}{2}\sqrt{\frac{\pi}{A}}=\sqrt{\frac{\pi}{A}},$$

$$\langle\psi|P^2|\psi\rangle=|P|\psi\rangle|^2=\int_{-\infty}^{\infty}\left|\frac{\hbar}{i}\frac{\partial}{\partial x}e^{-Ax^2/2}\right|^2dx=\int_{-\infty}^{\infty}\left|\hbar iAxe^{-Ax^2/2}\right|^2dx=2\hbar^2A^2\int_0^{\infty}x^2e^{-Ax^2}dx$$
$$=\frac{2\hbar^2A^2}{4}\sqrt{\frac{\pi}{A^3}}=\frac{1}{2}\hbar^2\sqrt{\pi A},$$

$$\langle\psi|V(x)|\psi\rangle=\frac{1}{2}m\left[\omega_2^2\int_{-\infty}^{0}x^2e^{-Ax^2}dx+\omega_1^2\int_0^{\infty}x^2e^{-Ax^2}dx\right]=\frac{1}{2}m\left(\omega_1^2+\omega_2^2\right)\int_0^{\infty}x^2e^{-Ax^2}dx$$
$$=\frac{1}{2}m\left(\omega_1^2+\omega_2^2\right)\frac{1}{4}\sqrt{\frac{\pi}{A^3}}.$$

We then just put everything together to get an expression for the energy expectation value as a function of A:

$$E(A)=\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}=\frac{1}{\langle\psi|\psi\rangle}\left[\frac{1}{2m}\langle\psi|P^2|\psi\rangle+\langle\psi|V(x)|\psi\rangle\right]$$
$$=\sqrt{\frac{A}{\pi}}\left[\frac{1}{2m}\frac{1}{2}\hbar^2\sqrt{\pi A}+\frac{1}{8}m\left(\omega_1^2+\omega_2^2\right)\sqrt{\frac{\pi}{A^3}}\right]=\frac{\hbar^2A}{4m}+\frac{m}{8A}\left(\omega_1^2+\omega_2^2\right).$$

We then minimize this by taking the derivative with respect to A and setting it equal to zero:

$$0=\frac{d}{dA}E(A)=\frac{\hbar^2}{4m}-\frac{m}{8A^2}\left(\omega_1^2+\omega_2^2\right),$$
$$A^2=\frac{m^2}{2\hbar^2}\left(\omega_1^2+\omega_2^2\right),$$
$$A=\frac{m}{\hbar}\sqrt{\tfrac{1}{2}\left(\omega_1^2+\omega_2^2\right)}.$$

We substitute this back into our formula for the energy to get an upper limit on the ground state energy:

$$E(A)=\frac{\hbar^2}{4m}\frac{m}{\hbar}\sqrt{\tfrac{1}{2}\left(\omega_1^2+\omega_2^2\right)}+\frac{m}{8}\frac{\hbar\left(\omega_1^2+\omega_2^2\right)}{m\sqrt{\tfrac{1}{2}\left(\omega_1^2+\omega_2^2\right)}}=\sqrt{\tfrac{1}{2}\left(\omega_1^2+\omega_2^2\right)}\left(\frac{\hbar}{4}+\frac{\hbar}{4}\right)=\tfrac{1}{2}\hbar\sqrt{\tfrac{1}{2}\left(\omega_1^2+\omega_2^2\right)}.$$

We note that if $\omega_1=\omega_2=\omega$, this simplifies to the correct formula $E=\frac{1}{2}\hbar\omega$.

4. The ground state of hydrogen is $\psi(r,\theta,\phi) = Ne^{-r/a}$.

(a) [5] What is the correct normalization N?

We demand the normalization integral yield 1; that is,

$$1 = \int |\psi|^2 d^3\mathbf{r} = |N|^2 \int_0^\infty e^{-2r/a} r^2 dr \int d\Omega = |N|^2 \left(\tfrac{1}{2}a\right)^3 2!(4\pi) = \pi a^3 |N|^2 ,$$

$$|N| = \frac{1}{\sqrt{\pi a^3}}.$$

(b) [13] What are the expectation values of $\langle Z \rangle$, $\langle Z^2 \rangle$, $\langle P_z \rangle$, and $\langle P_z^2 \rangle$?

We can save a bit of time by noticing that $\langle P_z \rangle$ automatically vanishes because the wave function is real, or we can simply do it. It is also helpful to notice that $\langle P_z^2 \rangle = \left| P_z |\psi\rangle \right|^2$. So we have

$$\langle Z \rangle = \langle \psi | Z | \psi \rangle = \frac{1}{\pi a^3} \int_0^\infty e^{-2r/a} r^2 r dr \int \cos\theta \, d\Omega = \frac{1}{\pi a^3} \left(\frac{a}{2}\right)^4 3!(0) = 0,$$

$$\langle Z^2 \rangle = \langle \psi | Z^2 | \psi \rangle = \frac{1}{\pi a^3} \int_0^\infty e^{-2r/a} r^2 r^2 dr \int \cos^2\theta \, d\Omega = \frac{1}{\pi a^3} \left(\frac{a}{2}\right)^5 4! \frac{4\pi}{3} = \frac{96 a^5}{96 a^3} = a^2 ,$$

$$\langle P_z \rangle = \langle \psi | P_z | \psi \rangle = \frac{1}{\pi a^3} \int_0^\infty e^{-r/a} r^2 dr \int \frac{\hbar}{i} \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \right) e^{-r/a} d\Omega$$

$$= \frac{i\hbar}{\pi a^4} \int_0^\infty e^{-2r/a} r^2 dr \int \cos\theta \, d\Omega = 0,$$

$$\langle P_z^2 \rangle = \left| P_z |\psi\rangle \right|^2 = \frac{1}{\pi a^3} \int_0^\infty r^2 dr \int \left| \frac{\hbar}{i} \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \right) e^{-r/a} \right|^2 d\Omega$$

$$= \frac{\hbar^2}{\pi a^5} \int_0^\infty e^{-2r/a} r^2 dr \int \cos^2\theta \, d\Omega = \frac{\hbar^2}{\pi a^5} \left(\frac{a}{2}\right)^3 2! \frac{4\pi}{3} = \frac{\hbar^2 a^3 8}{24 a^5} = \frac{\hbar^2}{3a^2}.$$

(c) [7] Find the uncertainties Δz and Δp_z and check that they satisfy the uncertainty relation.

These uncertainties are calculated in the standard way:

$$\Delta z = \sqrt{\langle Z^2 \rangle - \langle Z \rangle^2} = \sqrt{a^2 - 0^2} = a,$$

$$\Delta p_z = \sqrt{\langle P_z^2 \rangle - \langle P_z \rangle^2} = \sqrt{\frac{\hbar^2}{3a^2} - 0^2} = \frac{\hbar}{a\sqrt{3}},$$

$$(\Delta z)(\Delta p_z) = \tfrac{1}{\sqrt{3}}\hbar > \tfrac{1}{2}\hbar.$$

5. **A particle of mass *m* in 3D is in a potential** $V(x,y,z)=\begin{cases}\frac{1}{2}m\omega^2\left(x^2+y^2\right) & \text{for } |z|<a, \\ \infty & \text{for } |z|>a.\end{cases}$

   (a) [10] Write a formula for the energy of all the eigenstates of this potential. Also, write the wave function explicitly for the ground state. You do not need to normalize it.

The problem is basically separable in Cartesian coordinates, so we can simply solve it in the three dimensions separately. In the *x* and *y*-directions, it is a harmonic oscillator with frequency ω, so the corresponding energies are $E_x=\hbar\omega\left(n_x+\frac{1}{2}\right)$ and $E_y=\hbar\omega\left(n_y+\frac{1}{2}\right)$, where n_x and n_y are non-negative integers. For the *z*-direction, it is an infinite square well potential of width 2*a*, so it has energies $E_z=\dfrac{\pi^2\hbar^2}{2m(2a)^2}n_z^2$, but this time n_z is a *positive* integer. The total energy is

$$E_{n_x,n_y,n_z}=\hbar\omega\left(n_x+n_y+1\right)+\frac{\pi^2\hbar^2}{8ma^2}n_z^2\ ,\quad n_x,n_y=0,1,2,\ldots,\quad n_z=1,2,3,\ldots$$

For the ground state, we pick $n_x=n_y=0$ and $n_z=1$. The wave function for the ground state of the harmonic oscillator is given. For the infinite square well, the eigenstates are sines or cosines. Because we have a symmetric potential, the ground state will be a cosine, and because we want it to vanish at $z=\pm a$, we choose $\cos(\pi z/2a)$. Putting it all together, the ground state is

$$\psi_{0,0,1}(x,y,z)=N\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+y^2\right)\right]\cos\left(\frac{\pi z}{2a}\right).$$

The normalization is not requested, but it is pretty easy to perform the relevant integrals in cylindrical coordinates and find $N=\sqrt{\dfrac{\hbar}{\pi m\omega a}}$.

   (b) [8] It is found that the first excited state is triply degenerate; that is, there are three states with the same energy. From this, deduce a relationship between *m*, *a*, and ω.

The first excited state will be when one of n_x, n_y, n_z is increased by one. The only way to make the three states have the same energy is if all of them have the same energy. In other words, we must have

$$E_{101}=E_{011}=E_{002}$$

$$\hbar\omega(1+0+1)+\frac{\pi^2\hbar^2 1^2}{8ma^2}=\hbar\omega(0+1+1)+\frac{\pi^2\hbar^2 1^2}{8ma^2}=\hbar\omega(0+0+1)+\frac{\pi^2\hbar^2 2^2}{8ma^2},$$

$$\hbar\omega=\frac{\pi^2\hbar^2\left(2^2-1^2\right)}{8ma^2},$$

$$\omega=\frac{3\pi^2\hbar}{8ma^2}.$$

(c) [7] To this potential is added a small additional potential $W = \lambda YZ$. What effect will this have on the energies of the triply degenerate state to first order? In particular, write down the relevant 3×3 perturbation matrix, and determine which components must vanish, You do not need to calculate the non-vanishing components, nor must you do anything else with them.

Because the states are degenerate, we must use degenerate perturbation theory. The states look like $\left|n_x, n_y, n_z\right\rangle$, where the first two are harmonic oscillator states and the third one is an infinite square well state. We must then find all matrix elements of the form

$$\left\langle n_x, n_y, n_z \right| W \left| n_x', n_y', n_z' \right\rangle = \lambda \left\langle n_x, n_y, n_z \right| YZ \left| n_x', n_y', n_z' \right\rangle.$$

To save time, we note that the value of n_x must remain unchanged, while the Y operator can only raise or lower n_y by one. The three states we are working with are $\{|1,0,1\rangle, |0,1,1\rangle, |0,0,2\rangle\}$, and the only combinations that leave n_x unchanged while changing n_y by one are when we choose the last two. Let us define

$$A = \langle 0,1,1|YZ|0,0,2\rangle = \langle 0,0,2|YZ|0,1,1\rangle.$$

These are the only non-vanishing components. So the perturbation matrix is

$$\tilde{W} = \lambda A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We are now done with the problem as stated, but it is pretty easy to see that the three eigenstates will be $|1,0,1\rangle$ and $\frac{1}{\sqrt{2}}\left(|0,1,1\rangle \pm |0,0,2\rangle\right)$, and the energy shifts will be $\varepsilon' = 0$ for the first one and $\varepsilon' = \pm\lambda A$ for the other two. It remains only to work out A, which is more work.

<u>1D Harmonic Oscillator:</u> Ground state: $\psi_0(x) \propto \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$

<u>Spherical Coordinates:</u>

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \quad \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta},$$

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} - \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}, \quad \frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} - \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

<u>Possibly Helpful Integrals</u>

$$\int d\Omega = 4\pi, \quad \int\cos\theta\, d\Omega = 0, \quad \int\cos^2\theta\, d\Omega = \frac{4\pi}{3}, \quad \int\sin\theta\, d\Omega = \pi^2, \quad \int\sin^2\theta\, d\Omega = \frac{8\pi}{3}.$$

$$\int_0^\infty r^n e^{-r/b}dr = b^{n+1}n!, \quad \int_0^\infty e^{-Bx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{B}}, \quad \int_0^\infty xe^{-Bx^2}dx = \frac{1}{2B}, \quad \int_0^\infty x^2e^{-Bx^2}dx = \frac{1}{4}\sqrt{\frac{\pi}{B^3}}.$$