Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 10

1. [15] Two spinless particles are in the 1D infinite square well with allowed region $0<x<a$ are in one of the two states $|\psi\rangle = N\left(|\phi_1,\phi_2\rangle \pm |\phi_2,\phi_1\rangle\right)$, where N is a constant and $|\phi_n\rangle$ is the n'th eigenstate of the 1D infinite square well.

(a) Determine the normalization constant N and write explicitly the wave function $\psi(x_1,x_2)$.

To get the normalization right, we simply demand that

$$1=\langle\psi|\psi\rangle = N^2\left(\langle\phi_1,\phi_2|\pm\langle\phi_2,\phi_1|\right)\left(|\phi_1,\phi_2\rangle\pm|\phi_2,\phi_1\rangle\right)=N^2\left(1\pm 0\pm 0+1\right)=2N^2\ ,$$

from which we conclude that $N=\frac{1}{\sqrt{2}}$. The wave function, therefore, is

$$\begin{aligned}\psi(x_1,x_2)&=\langle x_1,x_2|\psi\rangle=\tfrac{1}{\sqrt{2}}\left(\langle x_1,x_2|\phi_1,\phi_2\rangle\pm\langle x_1,x_2|\phi_2,\phi_1\rangle\right)=\tfrac{1}{\sqrt{2}}\left[\phi_1(x_1)\phi_2(x_2)\pm\phi_2(x_1)\phi_1(x_2)\right]\\&=\frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)\pm\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right].\end{aligned}$$

(b) The positions of the two particles are measured. What is the probability that (i) $0<x_1<\frac{1}{2}a$, (ii) $0<x_2<\frac{1}{2}a$ and (iii) both are true.

We integrate the square of the wave function over the indicated range, with any unspecified integrals over the entire range from 0 to a. The range will depend on which part we are doing. The resulting integrals are given as Eqs. (A.12n) and (A.12q) in the appendix. We will do the indefinite integrals first, and then just substitute the appropriate limits.

$$\begin{aligned}\int dx_1\int dx_2\left|\psi(x_1,x_2)\right|^2&=\frac{2}{a^2}\int dx_1\int dx_2\left[\sin(\pi x_1/a)\sin(2\pi x_2/a)\pm\sin(2\pi x_1/a)\sin(\pi x_2/a)\right]^2\\&=\frac{2}{a^2}\int dx_1\int dx_2\begin{bmatrix}\sin^2(\pi x_1/a)\sin^2(2\pi x_2/a)+\sin^2(2\pi x_1/a)\sin^2(\pi x_2/a)\\ \pm 2\sin(\pi x_1/a)\sin(2\pi x_1/a)\sin(\pi x_2/a)\sin(2\pi x_2/a)\end{bmatrix}\\&=2\left[\tfrac{1}{2a}x_1-\tfrac{1}{4\pi}\sin(2\pi x_1/a)\right]\left[\tfrac{1}{2a}x_2-\tfrac{1}{8\pi}\sin(4\pi x_2/a)\right]\\&\quad+2\left[\tfrac{1}{2a}x_1-\tfrac{1}{8\pi}\sin(4\pi x_1/a)\right]\left[\tfrac{1}{2a}x_2-\tfrac{1}{4\pi}\sin(2\pi x_2/a)\right]\\&\quad\pm 4\left[\tfrac{1}{2\pi}\sin(\pi x_1/a)-\tfrac{1}{6\pi}\sin(3\pi x_1/a)\right]\left[\tfrac{1}{2\pi}\sin(\pi x_2/a)-\tfrac{1}{6\pi}\sin(3\pi x_2/a)\right]\end{aligned}$$

Substituting in our limits, we note that every term vanishes when $x_1=0$ or $x_2=0$, so we need only include the upper limits. We therefore have:

$$P\left(x_1<\tfrac{1}{2}a\right)=\int_0^{\frac{1}{2}a}dx_1\int_0^a dx_2\left|\psi(x_1,x_2)\right|^2=2\cdot\tfrac{1}{4}\cdot\tfrac{1}{2}+2\cdot\tfrac{1}{4}\cdot\tfrac{1}{2}\pm 4\left[\tfrac{1}{2\pi}\cdot 1-\tfrac{1}{6\pi}\cdot(-1)\right]\left[0-0\right]=\tfrac{1}{2}=50\%\ ,$$

$$P\left(x_2 < \tfrac{1}{2}a\right) = \int_0^a dx_1 \int_0^{\frac{1}{2}a} dx_2 \left|\psi\left(x_1, x_2\right)\right|^2 = 2\cdot\tfrac{1}{2}\cdot\tfrac{1}{4} + 2\cdot\tfrac{1}{2}\cdot\tfrac{1}{4} \pm 4\left[0-0\right]\left[\tfrac{1}{2\pi}\cdot 1 - \tfrac{1}{6\pi}\cdot\left(-1\right)\right] = \tfrac{1}{2} = 50\%\,,$$

$$P\left(x_1, x_2 < \tfrac{1}{2}a\right) = \int_0^{\frac{1}{2}a} dx_1 \int_0^{\frac{1}{2}a} dx_2 \left|\psi\left(x_1, x_2\right)\right|^2 = 2\left[\tfrac{1}{4}\right]^2 + 2\left[\tfrac{1}{4}\right]^2 \pm 4\left[\tfrac{1}{2\pi}\cdot 1 - \tfrac{1}{6\pi}\cdot\left(-1\right)\right]^2 = \tfrac{1}{4} \pm 4\left(\tfrac{2}{3\pi}\right)^2$$

$$= \frac{1}{4} \pm \frac{16}{9\pi^2} = \begin{cases} 43.01\% & \text{for } +, \\ 6.99\% & \text{for } -. \end{cases}$$