Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 10

2. [10] A system of more than two particles is an eigenstate of every pair switching operator, that is, $P(i \leftrightarrow j)|\psi\rangle = \lambda_{ij}|\psi\rangle$ for every $i \neq j$.

(a) [3] For *i*, *j*, and *k* all different, simplify the product $P(i \leftrightarrow j)P(i \leftrightarrow k)P(i \leftrightarrow j)$.

If we start on the right, we see that *i* goes to *j*, then it is left alone, then it goes to *i* again. We see that *j* goes to *i*, and then to *k*, and there it stays. Finally, *k* doesn't go anywhere at first, then it goes to *i*, and from there to *j*. Putting it all together, we see that

$$P(i \leftrightarrow j)P(i \leftrightarrow k)P(i \leftrightarrow j) = P(j \leftrightarrow k)$$

(b) [4] Demonstrate that $\lambda_{ik} = \lambda_{jk}$ for any *i*, *j*, and *k* all different.

We assume that our wave function is an eigenstate of all pair switching operators. We have

$$\begin{aligned}(i \leftrightarrow j)P(i \leftrightarrow k)P(i \leftrightarrow j)|\psi\rangle &= P(j \leftrightarrow k)|\psi\rangle \\ \lambda_{ij}\lambda_{ik}\lambda_{ij}|\psi\rangle &= \lambda_{jk}|\psi\rangle \\ \lambda_{ik}\lambda_{ij}^2 &= \lambda_{jk}\end{aligned}$$

We know that if we do a pair switching twice, you end up back where you started, which tells you $\lambda_{ij}^2 = 1$, so $\lambda_{ik} = \lambda_{jk}$.

(c) [3] Argue that for any pair λ_{ij} and λ_{kl}, $\lambda_{ij} = \lambda_{kl}$, whether there are any matches or not. Hence there is only one common eigenvalue for all pair switchings.

The order on the subscripts doesn't matter, $\lambda_{ij} = \lambda_{ji}$ since they represent the same pair switching. There are three cases: no indices match, one index matches, and both indices match. If both match, we have $\lambda_{ij} = \lambda_{ij}$, which is trivial. If one matches, we have $\lambda_{ik} = \lambda_{jk}$ from part (b). If neither matches, then we can argue that $\lambda_{ij} = \lambda_{il} = \lambda_{kl}$, since the intermediate one matches one index in each case. So we're done!

3. [15] Three particles lie in identical harmonic oscillators, with Hamiltonian

$$H = \sum_{i=1}^{3} \left(P_i^2 / 2m + \tfrac{1}{2} m\omega^2 X_i^2 \right)$$

(a) [5] If the particles are not identical, find the three lowest energies, give me a list of all quantum states having those energies, and tell me the degeneracy (number of states with that energy) for each.

E	states	#
$\frac{3}{2}\hbar\omega$	$\lvert 000\rangle$	1
$\frac{5}{2}\hbar\omega$	$\lvert 001\rangle, \lvert 010\rangle, \lvert 100\rangle$	3
$\frac{7}{2}\hbar\omega$	$\lvert 011\rangle, \lvert 101\rangle, \lvert 110\rangle,$ $\lvert 002\rangle, \lvert 020\rangle, \lvert 200\rangle$	6

Each harmonic oscillator has an energy of $E_i = \hbar\omega\left(n_i + \frac{1}{2}\right)$, so for the three harmonic oscillators together, the energy is

$$E = \hbar\omega\left(n_1 + n_2 + n_3 + \tfrac{3}{2}\right)$$

for the state $\lvert n_1 n_2 n_3 \rangle$ The smallest energies come when $n_1 + n_2 + n_3$ is as small as possible, or 0, 1, or 2. The table at right gives the full answer to this question. The "#" tells the degeneracy.

(b) [5] Repeat part (a) if the three particles are all bosons. I still want three energies, but I am not guaranteeing that the energies will be the same.

E	states	#
$\frac{3}{2}\hbar\omega$	$\lvert 000\rangle$	1
$\frac{5}{2}\hbar\omega$	$\frac{1}{\sqrt{3}}\left(\lvert 001\rangle + \lvert 010\rangle + \lvert 100\rangle\right)$	1
$\frac{7}{2}\hbar\omega$	$\frac{1}{\sqrt{3}}\left(\lvert 011\rangle + \lvert 101\rangle + \lvert 110\rangle\right),$ $\frac{1}{\sqrt{3}}\left(\lvert 002\rangle + \lvert 020\rangle + \lvert 200\rangle\right)$	2

The energies *will* be the same, we just have to symmetrize the wave functions. This reduces the degeneracy considerably.

(c) [5] Repeat part (a) if the three particles are all fermions.

E	states	#
$\frac{9}{2}\hbar\omega$	$\frac{1}{\sqrt{6}}\left(\lvert 012\rangle + \lvert 120\rangle + \lvert 201\rangle - \lvert 021\rangle - \lvert 210\rangle - \lvert 102\rangle\right)$	1
$\frac{11}{2}\hbar\omega$	$\frac{1}{\sqrt{6}}\left(\lvert 013\rangle + \lvert 130\rangle + \lvert 301\rangle - \lvert 031\rangle - \lvert 310\rangle - \lvert 103\rangle\right)$	1
$\frac{13}{2}\hbar\omega$	$\frac{1}{\sqrt{6}}\left(\lvert 023\rangle + \lvert 230\rangle + \lvert 302\rangle - \lvert 032\rangle - \lvert 320\rangle - \lvert 203\rangle\right),$ $\frac{1}{\sqrt{6}}\left(\lvert 014\rangle + \lvert 140\rangle + \lvert 401\rangle - \lvert 041\rangle - \lvert 410\rangle - \lvert 104\rangle\right)$	2

This time the ground state is the anti-symmetrized $\lvert 012\rangle$, which has a higher energy. We can then go up by one unit in $n_1 + n_2 + n_3$ to get the next two energies. It turns out only the third energy is degenerate.