Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 11

2. [10] At $t = 0$, the wave function of a free particle is given by

$$\Psi(x,t=0) = (A/\pi)^{1/4} \exp\left[-\tfrac{1}{2}A(x-b)^2\right]$$

Find $\Psi(x,t)$, using the propagator.

This is straightforward, though boring. The answer is

$$\Psi(x,t) = \int dx_0 K(x,t;x_0,0)\Psi(x_0,0)$$

$$= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \int dx_0 \exp\left[\frac{im(x-x_0)^2}{2\hbar t}\right] \exp\left[-\tfrac{1}{2}A(x_0-b)^2\right]$$

$$= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \int dx_0 \exp\left[\left(\frac{im}{2\hbar t}-\frac{A}{2}\right)x_0^2 + \left(-\frac{imx}{\hbar t}+Ab\right)x_0 + \frac{imx^2}{2\hbar t} - \frac{Ab^2}{2}\right]$$

$$= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \sqrt{\frac{\pi}{A/2 - im/2\hbar t}} \exp\left[\frac{(-imx/\hbar t + Ab)^2}{4(A/2 - im/2\hbar t)}\right] \exp\left[\frac{imx^2}{2\hbar t} - \frac{Ab^2}{2}\right]$$

$$= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{A\hbar t i + m}} \exp\left[\frac{A^2b^2\hbar^2t^2 - 2iAmxb\hbar t - m^2x^2}{2(A\hbar t - im)\hbar t} + \frac{imx^2 - \hbar tAb^2}{2\hbar t}\right]$$

$$= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{A\hbar t i + m}} \exp\left[\frac{A^2b^2\hbar^2t^2 - 2iAmxb\hbar t - m^2x^2}{2(A\hbar t - im)\hbar t} + \frac{(imx^2 - \hbar tAb^2)(A\hbar t - im)}{2(A\hbar t - im)\hbar t}\right]$$

$$= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{A\hbar t i + m}} \exp\left[\frac{Ab^2\hbar t im - 2iAmxb\hbar t + imA\hbar t x^2}{2(A\hbar t - im)\hbar t}\right]$$

$$= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{m + A\hbar t i}} \exp\left[\frac{imA(x-b)^2}{2(A\hbar t - im)}\right] = \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{m + A\hbar t i}} \exp\left[-\frac{mA(x-b)^2}{2(m + A\hbar t i)}\right]$$

The final form makes it clear at least that if $t = 0$, the wave function matches the initial wave function. Though it is less obvious, the wave function will spread out over time.