Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 11

3. [25] This problem should be worked entirely in the Heisenberg formulation of quantum mechanics. A particle lies in the one-dimensional harmonic oscillator potential, so $H = P^2/2m + \frac{1}{2}m\omega^2 X^2$.

(a) [5] Work out dX/dt and dP/dt.

According to the Heisenberg equations of motion,

$$\frac{dX}{dt} = \frac{i}{\hbar}[H,X] = \frac{i}{2m\hbar}\left[P^2, X\right] = \frac{i}{2m\hbar}\left(-i\hbar P - i\hbar P\right) = \frac{P}{m},$$
$$\frac{dP}{dt} = \frac{i}{\hbar}[H,P] = \frac{im\omega^2}{2\hbar}\left[X^2, P\right] = \frac{im\omega^2}{2\hbar}\left(i\hbar X + i\hbar X\right) = -m\omega^2 X.$$

(b) [5] Define the operators $a(t) = \sqrt{m\omega/2\hbar}\,X(t) + i\,P(t)/\sqrt{2\hbar m\omega}$ and its Hermitian conjugate $a^\dagger(t)$. Show that these satisfy equations $da(t)/dt \propto a(t)$ and $da^\dagger(t)/dt \propto a^\dagger(t)$, and determine the proportionality constant in each case.

$$\frac{da}{dt} = \sqrt{\frac{m\omega}{2\hbar}}\frac{\partial X}{\partial t} + \frac{i}{\sqrt{2\hbar m\omega}}\frac{\partial P}{\partial t} = \sqrt{\frac{m\omega}{2\hbar}}\frac{P}{m} - \frac{i}{\sqrt{2\hbar m\omega}}m\omega^2 X = -i\omega\left[\sqrt{\frac{m\omega}{2\hbar}}X + \frac{i}{\sqrt{2\hbar m\omega}}P\right] = -i\omega a,$$
$$\frac{da^\dagger}{dt} = \sqrt{\frac{m\omega}{2\hbar}}\frac{\partial X}{\partial t} - \frac{i}{\sqrt{2\hbar m\omega}}\frac{\partial P}{\partial t} = \sqrt{\frac{m\omega}{2\hbar}}\frac{P}{m} + \frac{i}{\sqrt{2\hbar m\omega}}m\omega^2 X = i\omega\left[\sqrt{\frac{m\omega}{2\hbar}}X - \frac{i}{\sqrt{2\hbar m\omega}}P\right] = i\omega a^\dagger.$$

(c) [5] Solve the differential equations for $a(t)$ and $a^\dagger(t)$ in terms of $a(0)$ and $a^\dagger(0)$. As a check, confirm that the Hamiltonian $H = \hbar\omega\left[a^\dagger(t)a(t) + \frac{1}{2}\right]$, is independent of time.

The solutions of $\frac{d}{dt}a(t) = -i\omega a(t)$ and $\frac{d}{dt}a^\dagger(t) = i\omega a^\dagger(t)$ are respectively

$$a(t) = e^{-i\omega t}a(0) \quad \text{and} \quad a^\dagger(t) = e^{i\omega t}a^\dagger(0).$$

Plugging these into the Hamiltonian, we see that the time dependence goes away.

$$H = \hbar\omega\left[a^\dagger(t)a(t) + \tfrac{1}{2}\right] = \hbar\omega\left[e^{i\omega t}a^\dagger(0)e^{-i\omega t}a(0) + \tfrac{1}{2}\right] = \hbar\omega\left[a^\dagger(0)a(0) + \tfrac{1}{2}\right].$$

(d) [5] Rewrite *X*(*t*) and *P*(*t*) in terms of $a(t)$ and $a^\dagger(t)$, and rewrite $a(0)$ and $a^\dagger(0)$ in terms of *X*(0) and *P*(0), so that *X*(*t*) and *P*(*t*) depend only on *X*(0) and *P*(0). You may find the identities below useful.

$$X(t)=\sqrt{\hbar/2m\omega}\left[a(t)+a^\dagger(t)\right] \quad \text{and} \quad P(t)=i\sqrt{\hbar m\omega/2}\left[a^\dagger(t)-a(t)\right].$$

As a check, you should find $X(T)=X(0)$, if *T* is the classical period.

These are fairly straightforward. We start with the position operator:

$$\begin{aligned}
X(t)&=\sqrt{\frac{\hbar}{2m\omega}}\left[a(t)+a^\dagger(t)\right]=\sqrt{\frac{\hbar}{2m\omega}}\left[e^{-i\omega t}a(0)+e^{i\omega t}a^\dagger(0)\right]\\
&=\sqrt{\frac{\hbar}{2m\omega}}\left[e^{-i\omega t}\left(\sqrt{\frac{m\omega}{2\hbar}}X(0)+\frac{i}{\sqrt{2\hbar m\omega}}P(0)\right)+e^{i\omega t}\left(\sqrt{\frac{m\omega}{2\hbar}}X(0)-\frac{i}{\sqrt{2\hbar m\omega}}P(0)\right)\right]\\
&=\frac{1}{2}X(0)\left(e^{-i\omega t}+e^{i\omega t}\right)+\frac{i}{2m\omega}P(0)\left(e^{-i\omega t}-e^{i\omega t}\right)=X(0)\cos(\omega t)+\frac{P(0)}{m\omega}\sin(\omega t).
\end{aligned}$$

We now do the momentum operator in exactly the same way.

$$\begin{aligned}
P(t)&=i\sqrt{\frac{\hbar m\omega}{2}}\left[a^\dagger(t)-a(t)\right]=i\sqrt{\frac{\hbar m\omega}{2}}\left[e^{i\omega t}a^\dagger(0)-e^{-i\omega t}a(0)\right]\\
&=i\sqrt{\frac{\hbar m\omega}{2}}\left[e^{i\omega t}\left(\sqrt{\frac{m\omega}{2\hbar}}X(0)-\frac{i}{\sqrt{2\hbar m\omega}}P(0)\right)-e^{-i\omega t}\left(\sqrt{\frac{m\omega}{2\hbar}}X(0)+\frac{i}{\sqrt{2\hbar m\omega}}P(0)\right)\right]\\
&=i\frac{m\omega}{2}X(0)\left(e^{i\omega t}-e^{-i\omega t}\right)+\frac{1}{2}P(0)\left(e^{i\omega t}+e^{-i\omega t}\right)=P(0)\cos(\omega t)-m\omega X(0)\sin(\omega t).
\end{aligned}$$

It is now obvious that if we set $T=2\pi/\omega$, the classical period, then $X(T)=X(0)$.

(e) [5] Suppose the quantum state (which is independent of time) is chosen to be an eigenstate of $X(0)$, $X(0)|\psi\rangle=x_0|\psi\rangle$. Show that at each of the times $t=\frac{1}{4}T$, $t=\frac{1}{2}T$, $t=\frac{3}{4}T$, and $t=T$, it is an eigenstate of either $X(t)$ or $P(t)$, and determine its eigenvalue.

These times correspond to $\omega t=\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π respectively. We therefore have

$$\begin{aligned}
&X\left(\tfrac{1}{4}T\right)=P(0)/m\omega, \quad X\left(\tfrac{1}{2}T\right)=-X(0), \quad X\left(\tfrac{3}{4}T\right)=-P(0)/m\omega, \quad X(T)=X(0),\\
&P\left(\tfrac{1}{4}T\right)=-m\omega X(0), \quad P\left(\tfrac{1}{2}T\right)=-P(0), \quad P\left(\tfrac{3}{4}T\right)=m\omega X(0), \qquad P(T)=P(0).
\end{aligned}$$

From these it is easy to see that

$$\begin{aligned}
P\left(\tfrac{1}{4}T\right)|\psi\rangle&=-m\omega x_0|\psi\rangle, \quad X\left(\tfrac{1}{2}T\right)|\psi\rangle=-x_0|\psi\rangle,\\
P\left(\tfrac{3}{4}T\right)|\psi\rangle&=m\omega x_0|\psi\rangle, \quad X(T)|\psi\rangle=x_0|\psi\rangle.
\end{aligned}$$