Solutions to Chapter 11

6. [20] There is another version of the disproof of the "local hidden variables" hypothesis that does *not* require a discussion of probabilities. Consider a system consisting of *three* spins in the state

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}\left(|+++\rangle - |---\rangle\right)$$

Each of these spins will be measured on either the *x*-axis or the *y*-axis, that is, we will be measuring one of each of the following pairs: $\left\{(\sigma_{x1},\sigma_{y1}),(\sigma_{x2},\sigma_{y2}),(\sigma_{x3},\sigma_{y3})\right\}$. The measurements will yield three eigenvalues, which will be one each from the pairs (x_1,y_1), (x_2,y_2), (x_3,y_3). Each of these eigenvalues can take only the values ±1.

(a) [6] Consider each of the operators

$$B_1 = \sigma_{x1}\sigma_{y2}\sigma_{y3}, \quad B_2 = \sigma_{y1}\sigma_{x2}\sigma_{y3}, \quad B_3 = \sigma_{y1}\sigma_{y2}\sigma_{x3}.$$

Show that $|\psi\rangle$ is an eigenstate of each of these operators, and calculate the eigenvalue.

We recall that $\sigma_x|\pm\rangle = |\mp\rangle$ and $\sigma_y|\pm\rangle = \pm i|\mp\rangle$, so

$$\begin{aligned}
B_1|\psi\rangle &= \tfrac{1}{\sqrt{2}}\sigma_{x1}\sigma_{y2}\sigma_{y3}\left(|+++\rangle - |---\rangle\right) = \tfrac{1}{\sqrt{2}}\left(\sigma_{x1}\sigma_{y2}\sigma_{y3}|+++\rangle - \sigma_{x1}\sigma_{y2}\sigma_{y3}|---\rangle\right) \\
&= \tfrac{1}{\sqrt{2}}\left(i^2|---\rangle - (-i)^2|+++\rangle\right) = \tfrac{1}{\sqrt{2}}\left(|+++\rangle - |---\rangle\right) = |\psi\rangle, \\
B_2|\psi\rangle &= \tfrac{1}{\sqrt{2}}\sigma_{y1}\sigma_{x2}\sigma_{y3}\left(|+++\rangle - |---\rangle\right) = \tfrac{1}{\sqrt{2}}\left(\sigma_{y1}\sigma_{x2}\sigma_{y3}|+++\rangle - \sigma_{y1}\sigma_{x2}\sigma_{y3}|---\rangle\right) \\
&= \tfrac{1}{\sqrt{2}}\left(i^2|---\rangle - (-i)^2|+++\rangle\right) = \tfrac{1}{\sqrt{2}}\left(|+++\rangle - |---\rangle\right) = |\psi\rangle, \\
B_1|\psi\rangle &= \tfrac{1}{\sqrt{2}}\sigma_{y1}\sigma_{y2}\sigma_{x3}\left(|+++\rangle - |---\rangle\right) = \tfrac{1}{\sqrt{2}}\left(\sigma_{y1}\sigma_{y2}\sigma_{x3}|+++\rangle - \sigma_{y1}\sigma_{y2}\sigma_{x3}|---\rangle\right) \\
&= \tfrac{1}{\sqrt{2}}\left(i^2|---\rangle - (-i)^2|+++\rangle\right) = \tfrac{1}{\sqrt{2}}\left(|+++\rangle - |---\rangle\right) = |\psi\rangle.
\end{aligned}$$

In other words, it is an eigenstate of all three operators with eigenvalue $+1$ in each case.

(b) [3] According to quantum mechanics, suppose you happened to measure one of the three combinations *B*1, *B*2, or *B*3. What is the prediction for the *product* of the results of those measurements, *x*1*y*2*y*3, *y*1*x*2*y*3, or *y*1*y*2*x*3?

Well, if you measure B_1, for example, you would need to measure σ_{x1}, σ_{y2}, and σ_{y3}, and you would get the three eigenvalues x_1, y_2 and y_3. But we know the product of these operators has eigenvalue +1 when you multiply them. A similar argument goes for the other two. So in summary,

$$x_1y_2y_3 = y_1x_2y_3 = y_1y_2x_3 = 1$$

Of course, it is probably wrong to write all these equations together, since you can't measure every one of these for a single particle, according to quantum mechanics.

(c) [3] According to the philosophy of hidden variables, the values for the products you found in part (b) must be true, even if you don't make those exact measurements. Based on the three formulas you found in part (b), make a prediction for the product $x_1x_2x_3$ (hint: consider the product of the three possibilities in part (b)). *This has nothing to do with quantum mechanics!*

The argument is simple. Each of the values that you measure on the three particles clearly will equal ± 1. We have three products that equal one. If we multiply these three things together, we must still get 1. Hence

$$1 = (x_1 y_2 y_3)(y_1 x_2 y_3)(y_1 y_2 x_3) = x_1 x_2 x_3 y_1^2 y_2^2 y_3^2$$

Now, since each variable is ± 1, it is clear that $y_1^2 = y_2^2 = y_3^2 = 1$. Therefore $1 = x_1 x_2 x_3$.

This conclusion doesn't really mean anything in quantum mechanics. You can't talk about all these other variables unless you actually measure them, so no such a-priori conclusion can be reached.

(d) [3] According to the philosophy of hidden variables, the product $x_1x_2x_3$ must be what you would get if you performed all three measurements in the x-direction, and then multiplied them. Hence, the operator

$$A = \sigma_{x1}\sigma_{x2}\sigma_{x3}$$

will yield what value if you were to measure it, according to the philosophy of hidden variables?

If we set up the experiment to measure these three operators instead, the philosophy of hidden variables demands that the product of the three measured quantities x_1, x_2, and x_3 must come out to $x_1 x_2 x_3 = 1$.

(e) [5] Show that $|\psi\rangle$ is, in fact, an eigenstate of A, but it has the wrong eigenvalue according to the prediction of part (d).

Proving it is an eigenstate is easy:

$$\begin{aligned} A|\psi\rangle &= \tfrac{1}{\sqrt{2}}\sigma_{x1}\sigma_{x2}\sigma_{x3}\left(|+++\rangle - |---\rangle\right) = \tfrac{1}{\sqrt{2}}\left(\sigma_{x1}\sigma_{x2}\sigma_{x3}|+++\rangle - \sigma_{x1}\sigma_{x2}\sigma_{x3}|---\rangle\right) \\ &= \tfrac{1}{\sqrt{2}}\left(|---\rangle - |+++\rangle\right) = -|\psi\rangle. \end{aligned}$$

Hence quantum mechanics predicts that the result will be -1 if you perform this set of three measurements. Since this experiment has not been done, we don't know which of the predictions is correct, but everyone assumes quantum mechanics is right.