Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 12

6. [20] A particle of mass *m* lies in one dimension with Hamiltonian near the origin of $H = P^2/2m + \frac{1}{2}m\omega^2 X^2 + \gamma X^3$ where γ is very small.

(a) [8] Find the energy of *all* quantum states to second order in γ.

If we treat γ as zero, then we have a harmonic oscillator, which has eigenstates $|n\rangle$ and energies $\hbar\omega\left(n+\frac{1}{2}\right)$. Letting our perturbation act on an arbitrary state, we have

$$W|n\rangle = \gamma\left(\frac{\hbar}{2m\omega}\right)^{3/2}\left(a+a^\dagger\right)^3|n\rangle = \gamma\left(\frac{\hbar}{2m\omega}\right)^{3/2}\left(a+a^\dagger\right)^2\left(\sqrt{n+1}\,|n+1\rangle+\sqrt{n}\,|n-1\rangle\right)$$

$$= \gamma\left(\frac{\hbar}{2m\omega}\right)^{3/2}\left(a+a^\dagger\right)\left(\sqrt{(n+1)(n+2)}\,|n+2\rangle+(2n+1)|n\rangle+\sqrt{n(n-1)}\,|n-2\rangle\right)$$

$$= \gamma\left(\frac{\hbar}{2m\omega}\right)^{3/2}\left(\begin{array}{l}\sqrt{(n+1)(n+2)(n+3)}\,|n+3\rangle+(3n+3)\sqrt{n+1}\,|n+1\rangle\\ +3n\sqrt{n}\,|n-1\rangle+\sqrt{n(n-1)(n-2)}\,|n-3\rangle\end{array}\right)$$

The energies of our states, therefore, are

$$E_n = \varepsilon_n + \langle n|W|n\rangle + \sum_{m\neq n}\frac{\left|\langle m|W|n\rangle\right|^2}{\varepsilon_n-\varepsilon_m}$$

$$= \hbar\omega\left(n+\tfrac{1}{2}\right)+0+\frac{\left|\langle n-3|W|n\rangle\right|^2}{3\hbar\omega}+\frac{\left|\langle n-1|W|n\rangle\right|^2}{\hbar\omega}+\frac{\left|\langle n+1|W|n\rangle\right|^2}{-\hbar\omega}+\frac{\left|\langle n+3|W|n\rangle\right|^2}{-3\hbar\omega}$$

$$= \hbar\omega\left(n+\tfrac{1}{2}\right)+\gamma^2\left(\frac{\hbar}{2m\omega}\right)^3\frac{1}{\hbar\omega}\left[\begin{array}{l}\frac{1}{3}n(n-1)(n-2)+(3n)^2 n-(3n+3)^2(n+1)\\ -\frac{1}{3}(n+1)(n+2)(n+3)\end{array}\right]$$

$$= \hbar\omega\left(n+\tfrac{1}{2}\right)+\frac{\gamma^2\hbar^2}{8m^3\omega^4}\left(\tfrac{1}{3}n^3-n^2+\tfrac{2}{3}n+9n^3-9n^3-27n^2-27n-9-\tfrac{1}{3}n^3-2n^2-\tfrac{11}{3}n-2\right)$$

$$= \hbar\omega\left(n+\tfrac{1}{2}\right)-\frac{\gamma^2\hbar^2}{8m^3\omega^4}\left(30n^2+30n+11\right)$$

(b) [5] Find the eigenstates of *H* to first order in γ.

The states are given, to first order in γ, by

$$
\begin{aligned}
|\psi_n\rangle &= |n\rangle + \sum_{m\neq n} \frac{|m\rangle\langle n|W|p\rangle}{\varepsilon_n - \varepsilon_m} \\
&= |n\rangle + |n+3\rangle \frac{\langle n+3|W|n\rangle}{-3\hbar\omega} + |n+1\rangle \frac{\langle n+1|W|n\rangle}{-\hbar\omega} + |n-1\rangle \frac{\langle n-1|W|n\rangle}{\hbar\omega} + |n-3\rangle \frac{\langle n-3|W|n\rangle}{3\hbar\omega} \\
&= |n\rangle + \frac{\gamma}{3\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^{3/2} \Big(-\sqrt{(n+3)(n+2)(n+1)}\,|n+3\rangle - 9(n+1)^{3/2}|n+1\rangle + 9n^{3/2}|n-1\rangle \\
&\qquad + \sqrt{n(n-1)(n-2)}\,|n-3\rangle\Big)
\end{aligned}
$$

(c) [7] Find the expectation value $\langle\psi_n|X|\psi_n\rangle$ for the eigenstates $|\psi_n\rangle$ to first order in γ.

We simply take

$$
\begin{aligned}
\langle\psi_n|X|\psi_n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n| + \sum_{m\neq n}\frac{\langle n|W|m\rangle\langle m|}{\varepsilon_n-\varepsilon_m}\right](a+a^\dagger)\left[|n\rangle + \sum_{m\neq n}\frac{|m\rangle\langle m|W|n\rangle}{\varepsilon_n-\varepsilon_m}\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\sum_{m\neq n}\frac{1}{\varepsilon_n-\varepsilon_m}\left[\langle n|W|m\rangle\langle m|(a+a^\dagger)|n\rangle + \langle n|(a+a^\dagger)|m\rangle\langle m|W|n\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\frac{\sqrt{n}}{\hbar\omega}\big(\langle n|W|n-1\rangle + \langle n-1|W|n\rangle\big) + \frac{\sqrt{n+1}}{-\hbar\omega}\big(\langle n|W|n+1\rangle + \langle n+1|W|n\rangle\big)\right] \\
&= \frac{2\gamma}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}}\left(\frac{\hbar}{2m\omega}\right)^{3/2}\left[\sqrt{n}\,3n\sqrt{n} - \sqrt{n+1}\,(3n+3)\sqrt{n+1}\right] \\
&= \frac{6\gamma}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^2\left[n^2-(n+1)^2\right] = -\frac{3\gamma\hbar(2n+1)}{2m^2\omega^3}
\end{aligned}
$$

7. [15] A particle of mass *m* lies in a one-dimensional infinite slightly tilted square well

$$
V(x) = \begin{cases} bx & \text{if } 0 < x < a \\ \infty & \text{otherwise} \end{cases}
$$

where *b* is small.

(a) [2] What are the normalized eigenstates and energies if *b* = 0?

For $b = 0$, we just have an infinite square well, with wave functions and energies

$$
\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right)
$$

$$
\varepsilon_n = \frac{\pi^2\hbar^2 n^2}{2ma^2}
$$

(b) [6] Find the eigenstates and energies to first order in *b*.

The energies are given, to first order in *b*, by

$$E_n = \varepsilon_n + \langle \phi_n | W | \phi_n \rangle = \frac{\pi^2 \hbar^2 n^2}{2ma^2} + b\frac{2}{a}\int_0^a x \sin^2\left(\frac{\pi n x}{a}\right) dx = \frac{\pi^2 \hbar^2 n^2}{2ma^2} + b\frac{2}{a}\frac{a^2}{4} = \frac{\pi^2 \hbar^2 n^2}{2ma^2} + \frac{ba}{2}$$

To find the wave functions, we need to find

$$\langle \phi_p | W | \phi_n \rangle = b\frac{2}{a}\int_0^a x \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi p x}{a}\right) dx = \frac{4banp\left[(-1)^{n+p} - 1\right]}{\pi^2 \left(n^2 - p^2\right)^2}$$

We are avoiding the letter *m* so as not to confuse it with the mass. The expression in []'s vanishes when $n + p$ is even, and is – 2 when $n + p$ is odd. So we have

$$|\psi_n\rangle = |\phi_n\rangle + \sum_{p\neq n} |\phi_p\rangle \frac{\langle \phi_p | W | \phi_n \rangle}{\varepsilon_n - \varepsilon_p} = |\phi_n\rangle + \sum_{\substack{p+n \\ \text{odd}}} \frac{-8banp}{\pi^2\left(n^2-p^2\right)^2} \frac{2ma^2}{\pi^2\hbar^2\left(n^2-p^2\right)} |\phi_p\rangle$$

$$= |\phi_n\rangle - \frac{16bma^3}{\pi^4\hbar^2} \sum_{\substack{p+n \\ \text{odd}}} \frac{np}{\left(n^2-p^2\right)^3} |\phi_p\rangle .$$

(c) [7] Find the ground state energy to second order in *b*. If you can't do the sums exactly, do them numerically.

Using our formula, we have

$$E_1 = \varepsilon_1 + \langle \phi_1 | W | \phi_1 \rangle + \sum_{p\neq 1} \frac{\left|\langle \phi_p | W | \phi_1 \rangle\right|^2}{\varepsilon_1 - \varepsilon_p} = \frac{\pi^2\hbar^2}{2ma^2} + \frac{ba}{2} + \sum_{p \text{ even}} \left[\frac{-8bap}{\pi^2\left(1-p^2\right)^2}\right]^2 \frac{2ma^2}{\pi^2\hbar^2\left(1-p^2\right)}$$

$$= \frac{\pi^2\hbar^2}{2ma^2} + \frac{ba}{2} - \frac{128mb^2a^4}{\pi^6\hbar^2} \sum_{p \text{ even}} \frac{p^2}{\left(p^2-1\right)^5}$$

The final sum can be performed numerically very quickly, since the terms fall as p^{-8}, or we can do it analytically by hand or with the help of Maple.

```
> sum(4*n^2/(4*n^2-1)^5,n=1..infinity);
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$$E_1 = \frac{\pi^2\hbar^2}{2ma^2} + \frac{ba}{2} - \frac{128mb^2a^4}{\pi^6\hbar^2}\left(\frac{5\pi^2}{1024} - \frac{\pi^4}{3072}\right) = \frac{\pi^2\hbar^2}{2ma^2} + \frac{ba}{2} - \frac{\left(15-\pi^2\right)mb^2a^4}{24\pi^4\hbar^2}$$

$$= \frac{\pi^2\hbar^2}{2ma^2} + \frac{ba}{2} - 0.002195\frac{mb^2a^4}{\hbar^2}$$