Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 14

3. [15] A particle of mass μ impacts a stationary weak potential given by $V(\mathbf{r}) = Aze^{-\beta r^2/2}$.

(a) [4] Find the Fourier transform $\int d^3\mathbf{r}V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}$ for this potential for arbitrary K. Note that this potential is *not* rotationally symmetric, so you do not have freedom to pick K to be any axis you want. I found it easiest to work in Cartesian coordinates.

$$\int d^3\mathbf{r}V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}} = A\iiint dx\,dy\,dz\,z\exp\left[-\tfrac{1}{2}\beta\left(x^2+y^2+z^2\right)-iK_x x-iK_y y-iK_z z\right]$$
$$= A\sqrt{\frac{2\pi}{\beta}}e^{-K_x^2/(2\beta)}\sqrt{\frac{2\pi}{\beta}}e^{-K_y^2/(2\beta)}\sqrt{\frac{2\pi}{\beta}}i\frac{\partial}{\partial K_z}e^{-K_z^2/(2\beta)} = -iA(2\pi)^{3/2}K_z\beta^{-5/2}e^{-\mathbf{K}^2/(2\beta)}.$$

The three separate integrals in the three coordinate directions were accomplished with the help of equation (A.29) from the notes.

(b) [5] A particle with wave number k comes in along the z-axis. Find the differential cross section and the total cross section.

From the notes, $\mathbf{K} = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta - 1)$ and $\mathbf{K}^2 = 2k^2(1-\cos\theta)$. Therefore, the differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4}\left|\frac{A(2\pi)^{\frac{3}{2}}}{\beta^{\frac{5}{2}}}iK_z e^{-\mathbf{K}^2/(2\beta)}\right|^2 = \frac{2\pi A^2\mu^2k^2}{\hbar^4\beta^5}(1-\cos\theta)^2e^{-2k^2(1-\cos\theta)/\beta}.$$

The total cross-section is obtained by integrating this over the relevant angles, which yields

$$\sigma = \int\frac{d\sigma}{d\Omega}d\Omega = \frac{4\pi^2A^2\mu^2k^2}{\hbar^4\beta^5}\int_{-1}^{1}(1-\cos\theta)^2\exp\left[-2k^2(1-\cos\theta)/\beta\right]d(\cos\theta)$$
$$= \frac{4\pi^2A^2\mu^2k^2}{\hbar^4\beta^5}\left(\frac{\beta}{2k^2}\right)^3\int_0^{4k^2/\beta}w^2e^{-w}dw = \frac{\pi^2A^2\mu^2}{\hbar^4k^4\beta^2}\left[1-\left(1+\frac{4k^2}{\beta}+\frac{8k^4}{\beta^2}\right)e^{-4k^2/\beta}\right].$$

(c) [6] The same particle comes in along the x-axis. Find the differential cross section and the total cross section. You might find it useful to rotate the problem to make it easier.

We want to define the scattering problem with the angle θ describing the final angle compared to the initial angle. It is easiest to do this by rotating the problem so the incoming wave is still along the z-axis. The factor of z in the potential will then turn into a factor of x, or perhaps, - x, depending on which way we rotate it (you can also get it along y if you prefer, but why borrow trouble?). Hence the Fourier transform will be exactly the same, except that K_z will get replaced by K_x, or perhaps – K_x. Since the Fourier transform gets squared anyway, it won't make a difference which way we do it. The only change in the cross section is that the factor of $k(\cos\theta-1)$ will become $k\sin\theta\cos\phi$, so

$$\frac{d\sigma}{d\Omega}=\frac{2\pi A^2\mu^2k^2}{\hbar^4\beta^5}\sin^2\theta\cos^2\phi\exp\left[-2k^2\left(1-\cos\theta\right)/\beta\right].$$

The ϕ integral is pretty easy, but we have to completely redo the θ integral. We find

$$\begin{aligned}\sigma&=\int\frac{d\sigma}{d\Omega}d\Omega=\frac{2\pi A^2\mu^2k^2}{\hbar^4\beta^5}\int_0^{2\pi}\cos^2\phi\,d\phi\int_{-1}^{1}\sin^2\theta\exp\left[-2k^2\left(1-\cos\theta\right)/\beta\right]d\cos\theta\\&=\frac{2\pi^2A^2\mu^2k^2}{\hbar^4\beta^5}\int_{-1}^{1}\left(1-\cos\theta\right)\left(1+\cos\theta\right)\exp\left[-2k^2\left(1-\cos\theta\right)/\beta\right]d\cos\theta\\&=\frac{2\pi^2A^2\mu^2k^2}{\hbar^4\beta^5}\left(\frac{\beta}{2k^2}\right)^3\int_0^{4k^2/\beta}w\left(4k^2/\beta-w\right)e^{-w}dw=\frac{\pi^2A^2\mu^2}{2\hbar^4\beta^2k^4}\left[\frac{2k^2}{\beta}-1+\left(\frac{2k^2}{\beta}+1\right)\exp\left(-\frac{4k^2}{\beta}\right)\right].\end{aligned}$$