Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 14

4. [10] In class we found the scattering cross section for Coulomb scattering by a charge *q′* from a point charge *q* located at the origin, with potential $V(\mathbf{r}) = k_e qq'/r$. We needed the Fourier transform, which turned out to be $\int d^3\mathbf{r} V(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} = 4\pi k_e qq'/\mathbf{K}^2$.

(a) [3] Place the source *q* not at the origin, but at $\mathbf{r} = \mathbf{a}$. What is the potential now? What is the Fourier transform now? Hint: **Don't actually do the work, just shift your integration variable, and use previous work. Convince yourself (and me) that the differential cross-section is unchanged.**

We need to replace *r* in the denominator of the potential with $|\mathbf{r}-\mathbf{a}|$. We then simply shift our integration variable to $\mathbf{r} \to \mathbf{r}+\mathbf{a}$. We find

$$\int d^3\mathbf{r} V(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} = k_e qq' \int \frac{d^3\mathbf{r}}{|\mathbf{r}-\mathbf{a}|} e^{-i\mathbf{K}\cdot\mathbf{r}} = k_e qq' \int \frac{d^3\mathbf{r}}{|\mathbf{r}+\mathbf{a}-\mathbf{a}|} e^{-i\mathbf{K}\cdot(\mathbf{r}+\mathbf{a})} = \frac{4\pi k_e qq'}{\mathbf{K}^2} e^{-i\mathbf{K}\cdot\mathbf{a}}$$

The only change is a phase, but since you end up squaring the amplitude, it doesn't change things at all.

(b) [4] Suppose we replaced *q* with a series of charges q_i located at several locations $\mathbf{a}_i$. What would be the Fourier transform now? What if, instead of a series of discrete charges q_i, we had a charge distribution $\rho(\mathbf{r})$ spread around in space?

The potential from a series of charges is obviously

$$V(\mathbf{r}) = \sum_i \frac{k_e q_i q'}{|\mathbf{r}-\mathbf{a}_i|}.$$

The Fourier transform of this, as a consequence, is clearly just the sum from each of the separate charges

$$\int d^3\mathbf{r} V(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} = \frac{4\pi k_e q'}{\mathbf{K}^2} \sum_i q_i e^{-i\mathbf{K}\cdot\mathbf{a}_i} .$$

For a charge distribution, the obvious generalization is

$$\int d^3\mathbf{r} V(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} = \frac{4\pi k_e q'}{\mathbf{K}^2} \int d^3\mathbf{r}\, \rho(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} .$$

(c) [3] Show that the differential cross-section for a charge q' scattering off of a charge distribution $\rho(\mathbf{r})$ is given by

$$\frac{d\sigma}{d\Omega} = \frac{4\mu^2 k_e^2 q'^2}{\hbar^4 \left(\mathbf{K}^2\right)^2} \left| \int d^3\mathbf{r} \rho(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} \right|^2$$

where $\mathbf{K} = \mathbf{k}' - \mathbf{k}$, the change in the wave number.

We simply substitute our previous result into the formula for the cross-section, which yields

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4} \left| \frac{4\pi k_e q'}{\mathbf{K}^2} \int d^3\mathbf{r}\, \rho(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} \right|^2 = \frac{4\mu^2 k_e^2 q'^2}{\hbar^4 \left(\mathbf{K}^2\right)^2} \left| \int d^3\mathbf{r}\, \rho(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} \right|^2 .$$

5. **[15] An electron of mass *m* scatters from a neutral hydrogen atom in the ground state located at the origin.**
   (a) [7] What is the charge distribution for a neutral hydrogen atom? Don't forget the nucleus! What is the Fourier transform of the charge?

The nucleus has charge e and is locate at the origin, so we can model it with a charge distribution $\rho(\mathbf{r}) = e\delta^3(\mathbf{r})$. The electron is spread out in a wave function given by $\rho(\mathbf{r}) = -e|\psi(\mathbf{r})|^2$. Using the explicit form of the wave function for the ground state, we therefore have

$$\rho(\mathbf{r}) = e\delta^3(\mathbf{r}) - \frac{e}{\pi a_0^3} e^{-2r/a_0} .$$

We will have to keep our e's straight. When computing the Fourier transform, the charge distribution is spherically symmetric, so there's no harm in assuming **K** is in the z-direction. The Fourier transform of this is

$$\begin{aligned}
\int d^3\mathbf{r}\rho(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} &= e - \frac{e}{\pi a_0^3} \int d^3\mathbf{r}\, e^{-2r/a_0} e^{-i\mathbf{K}\cdot\mathbf{r}} = e - \frac{2e}{a_0^3} \int_{-1}^{1} d\cos\theta \int_0^\infty r^2 dr \exp\left(-2r/a_0 - iKr\cos\theta\right) \\
&= e - \frac{4e}{a_0^3} \int_{-1}^{1} \left(2/a_0 + iK\cos\theta\right)^{-3} d\cos\theta = e + \frac{2e}{iKa_0^3} \left(2/a_0 + iK\cos\theta\right)^{-2} \Big|_{\cos\theta=-1}^{\cos\theta=1} \\
&= e + \frac{2e}{iKa_0^3} \left[\frac{1}{\left(2/a_0 + iK\right)^2} - \frac{1}{\left(2/a_0 - iK\right)^2} \right] = e + \frac{2e}{iKa_0^3} \frac{-8iK/a_0}{\left(4/a_0^2 + K^2\right)^2} \\
&= e - \frac{16e}{\left(4 + K^2 a_0^2\right)^2} = e \frac{8K^2 a_0^2 + K^4 a_0^4}{\left(4 + K^2 a_0^2\right)^2} .
\end{aligned}$$

Obviously, this vanishes if $K = 0$.

(b) [8] Find the differential and total cross section in this case.

We simply use the formula from problem 4. We find

$$\frac{d\sigma}{d\Omega} = \frac{4m^2k_e^2q'^2}{\hbar^4\left(\mathbf{K}^2\right)^2}\left|\int d^3\mathbf{r}\,\rho(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2 = \frac{4m^2k_e^2e^4\left(8a_0^2+K^2a_0^4\right)^2}{\hbar^4\left(4+K^2a_0^2\right)^4} = a_0^2\frac{\left[4+k^2a_0^2\left(1-\cos\theta\right)\right]^2}{\left[2+k^2a_0^2\left(1-\cos\theta\right)\right]^4},$$

where we have used the identity $\mathbf{K}^2 = 2k^2\left(1-\cos\theta\right)$ to simplify our expression, as well as $a_0 = \hbar^2/mk_ee^2$. We now have a single nasty integral to do, which is why God invented Maple.

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> assume(A>0);integrate((4+A-A*x)^2/(2+A-A*x)^4,x=-1..1);
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$$\sigma = 2\pi a_0^2\int_{-1}^{1}\frac{\left[4+k^2a_0^2\left(1-\cos\theta\right)\right]^2}{\left[2+k^2a_0^2\left(1-\cos\theta\right)\right]^4}d\cos\theta$$

$$= 4\pi a_0^2\frac{1+\frac{3}{2}k^2a_0^2+\frac{7}{12}k^4a_0^4}{\left(1+k^2a_0^2\right)^3}$$

Note that the final factor goes to one in the low energy limit, so we have a finite cross section $4\pi a_0^2$, which then decreases as we increase our energy.