Physics 742 – Graduate Quantum Mechanics 2

Solutions to Chapter 15

3. [15] A particle of mass *m* is initially in the ground state of a harmonic oscillator with frequency ω. At time *t* = 0, a perturbation is suddenly turned on of the form $W(t) = AXe^{-\lambda t}$. At late times ($t \to \infty$), the quantum state is measured again.

(a) [10] Calculate, to second order in *A*, the amplitude S_{n0} that it ends up in the state $|n\rangle$, for all *n* (most of them will be zero).

We will need matrix elements of the form $\langle n|W(t)|m\rangle = A\langle n|X|m\rangle e^{-\lambda t}$, which are given by

$$Ae^{-\lambda t}\langle n|X|m\rangle = Ae^{-\lambda t}\sqrt{\frac{\hbar}{2m\omega}}\langle n|\left(a+a^\dagger\right)|m\rangle = Ae^{-\lambda t}\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{m}\delta_{n,m-1}+\sqrt{n}\delta_{m,n-1}\right).$$

To second order, then, our amplitudes are

$$S_{00} = \delta_{00} + \frac{1}{i\hbar}\int_0^\infty \langle 0|W(t)|0\rangle dt + \frac{1}{(i\hbar)^2}\sum_n \int_0^\infty \langle 0|W(t)|n\rangle e^{i\omega_{0n}t}dt\int_0^t \langle n|W(t')|0\rangle e^{i\omega_{n0}t}dt'$$

$$= 1 - \frac{A^2}{\hbar^2}\left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2\int_0^\infty e^{-\lambda t}e^{-i\omega t}dt\int_0^t e^{-\lambda t'}e^{i\omega t'}dt' = 1 - \frac{A^2}{2m\omega\hbar(-\lambda+i\omega)}\int_0^\infty e^{-\lambda t}e^{-i\omega t}\left[e^{-(\lambda-i\omega)t}-1\right]dt$$

$$= 1 + \frac{A^2}{4m\omega^2\hbar(\lambda-i\omega)}\left[\frac{1}{2\lambda}-\frac{1}{\lambda+i\omega}\right] = 1 - \frac{A^2(\lambda-i\omega)}{4m\omega\hbar\lambda\left(\lambda^2+\omega^2\right)},$$

$$S_{10} = \delta_{10} + \frac{1}{i\hbar}\int_0^\infty \langle 1|W(t)|0\rangle e^{i\omega_{10}t}dt + \frac{1}{(i\hbar)^2}\sum_n \int_0^\infty \langle 1|W(t)|n\rangle e^{i\omega_{1n}t}dt\int_0^t \langle n|W(t')|0\rangle e^{i\omega_{n0}t'}dt'$$

$$= \frac{A}{i\hbar}\sqrt{\frac{\hbar}{2m\omega}}\int_0^\infty e^{i\omega t-\lambda t}dt = \frac{A}{i\sqrt{2m\hbar\omega}(\lambda-i\omega)},$$

$$S_{20} = \delta_{20} + \frac{1}{i\hbar}\int_0^\infty \langle 2|W(t)|0\rangle e^{i\omega_{20}t}dt + \frac{1}{(i\hbar)^2}\sum_n \int_0^\infty \langle 2|W(t)|n\rangle e^{i\omega_{2n}t}dt\int_0^t \langle n|W(t')|0\rangle e^{i\omega_{n0}t'}dt'$$

$$= -\frac{A^2}{\hbar^2}\left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2\int_0^\infty \sqrt{2}e^{i\omega t-\lambda t}dt\int_0^t e^{i\omega t'-\lambda t'}dt' = -\frac{A^2}{\sqrt{2}\hbar m\omega(i\omega-\lambda)}\int_0^\infty e^{i\omega t-\lambda t}\left[e^{i\omega t-\lambda t}-1\right]dt$$

$$= \frac{A^2}{\sqrt{2}\hbar m\omega(\lambda-i\omega)}\left[\frac{1}{2(\lambda-i\omega)}-\frac{1}{\lambda-i\omega}\right] = -\frac{A^2}{2\sqrt{2}\hbar m\omega(\lambda-i\omega)^2},$$

$$S_{n0} = 0, \quad n > 2.$$

(b) [5] Calculate, to at least second order, the probability that it ends up in the state $|n\rangle$. Check that the sum of the probabilities is 1, to second order in A.

We now simply square the amplitudes we found previously, and we find

$$P(0\to 0)=|S_{00}|^2=\left[1-\frac{A^2(\lambda-i\omega)}{4m\omega\hbar\lambda(\lambda^2+\omega^2)}\right]\left[1-\frac{A^2(\lambda+i\omega)}{4m\omega\hbar\lambda(\lambda^2+\omega^2)}\right]\approx 1-\frac{A^2}{2m\omega\hbar(\lambda^2+\omega^2)},$$

$$P(0\to 1)=|S_{10}|^2=\frac{A^2}{2m\hbar\omega(\lambda^2+\omega^2)},$$

$$P(0\to 2)=|S_{20}|^2=\frac{A^4}{8\hbar^2m^2\omega^2(\lambda^2+\omega^2)^2}.$$

To second order in A, the first two expressions add to 1 and the third is zero.