

Physics 742 – Graduate Quantum Mechanics 2  
 Solutions to Chapter 15

4. [10] A particle of mass  $m$  is in the ground state  $|1\rangle$  of an infinite square well with allowed region  $0 < X < a$ . To this potential is added a harmonic perturbation  $W(t) = AX \cos(\omega t)$ , where  $A$  is small.
- (a) [6] Calculate the transition rate  $\Gamma(1 \rightarrow n)$  for a transition to another level. Don't let the presence of a delta function bother you. What angular frequency  $\omega$  is necessary to make the transition occur to level  $n = 2$ ?

We first rewrite this perturbation in the form  $W(t) = \frac{1}{2} AX (e^{i\omega t} + e^{-i\omega t})$ . Our interaction is thus  $W = \frac{1}{2} AX$ , and the matrix elements we require are  $W_{1n} = \frac{1}{2} A \langle n | X | 1 \rangle$ . The eigenstates and energies of the unperturbed square well are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right), \quad E_n = \frac{\pi^2 n^2 \hbar^2}{2ma^2}.$$

Maple is happy to do the integrals for us. We find the matrix elements we want are

$$W_{n1} = \frac{A}{2} \langle n | X | 1 \rangle = \frac{A}{2} \cdot \frac{2}{a} \int_0^a x \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi x}{a}\right) dx = -\frac{2Aan}{\pi^2 (n^2 - 1)^2} [1 + (-1)^n].$$

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> assume(n::integer);
> integrate(A/a*sin(Pi*n*x/a)*sin(Pi*x/a)*x,x=0..a);
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The rate then is given by

$$\Gamma(1 \rightarrow n) = \frac{2\pi}{\hbar} |W_{n1}|^2 \delta(E_n - E_1 - \hbar\omega) = \frac{8a^2 A^2 n^2 [1 + (-1)^n]^2}{\pi^3 \hbar (n^2 - 1)^4} \delta\left(\frac{\pi^2 \hbar^2}{2ma^2} (n^2 - 1) - \hbar\omega\right).$$

For the transition to  $n = 2$ , we need the delta function to vanish, which requires a frequency of  $\omega = 3\pi^2 \hbar / 2ma^2$ .

- (b) [4] Now, instead of keeping a constant frequency, the frequency is tuned continuously, so that at  $t = 0$  the frequency is 0, and it rises linearly so that at  $t = T$  it has increased to the value  $\omega(T) = 2\pi^2 \hbar / ma^2$ . The tuning is so slow that at any given time, we may treat it as a harmonic source. Argue that only the  $n = 2$  state can become populated (to leading order in  $A$ ). Calculate the probability of a transition using  $P(1 \rightarrow 2) = \int_0^T \Gamma(1 \rightarrow 2) dt$ .

The maximum frequency achieved is 4/3 times larger than required to make this transition, enough for the 1 to 2 transition, but not enough for anything higher. We therefore need only consider the transition to  $n = 2$ . The frequency as a function of time is

$$\omega(t) = \frac{2\pi^2 \hbar t}{ma^2 T}$$

To find the total transition probability, we then simply integrate the transition rate

$$P(1 \rightarrow 2) = \int_0^T \Gamma(1 \rightarrow 2) dt = \frac{128a^2 A^2}{81\pi^3 \hbar} \int_0^T \delta\left(\frac{3\pi^2 \hbar^2}{2ma^2} - \frac{2\pi^2 \hbar^2 t}{ma^2 T}\right) dt = \frac{128a^2 A^2}{81\pi^3 \hbar} \frac{ma^2 T}{2\pi^2 \hbar^2} = \frac{64ma^4 A^2 T}{81\pi^5 \hbar^3}.$$