Physics 742 – Graduate Quantum Mechanics 2

Solutions to Chapter 16

2. [15] In the class notes, we claimed that the spin was defined by $\mathbf{S} = \frac{1}{2}\hbar\mathbf{\Sigma}$, eq. (16.17). To make sure this is plausible:

a) [3] Demonstrate that S **satisfies the commutations relations $\left[S_i, S_j\right] = \sum_k i\hbar\varepsilon_{ijk} S_k$.**

Taking advantage of the commutation relations of the Pauli matrices, we have:

$$\left[S_i, S_j\right] = \frac{\hbar^2}{4}\left[\Sigma_i, \Sigma_j\right] = \frac{\hbar^2}{4}\left[\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}\begin{pmatrix}\sigma_j & 0\\ 0 & \sigma_j\end{pmatrix} - \begin{pmatrix}\sigma_j & 0\\ 0 & \sigma_j\end{pmatrix}\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}\right]$$

$$= \frac{\hbar^{2\,2}}{4}\begin{pmatrix}\left[\sigma_i, \sigma_j\right] & 0\\ 0 & \left[\sigma_i, \sigma_j\right]\end{pmatrix} = \frac{\hbar^2}{2}\sum_k i\varepsilon_{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix} = i\frac{\hbar^2}{2}\sum_k \varepsilon_{ijk}\Sigma_k = i\hbar\sum_k \varepsilon_{ijk} S_k .$$

b) [7] Work out the six commutators $\left[\mathbf{L}, H\right]$ and $\left[\mathbf{S}, H\right]$ for the free Dirac Hamiltonian.

The commutators of **L** are pretty straightforward to compute. It commutes with the matrices $\boldsymbol{\alpha}$ and β, and we know the commutation relations of **L** with the momentum operator, so

$$\left[L_i, H\right] = \left[L_i, c\boldsymbol{\alpha}\cdot\mathbf{P} + mc^2\beta\right] = \sum_j \left[L_i, c\alpha_j P_j\right] = \sum_j c\alpha_j\left[L_i, P_j\right] = i\hbar c\sum_j\sum_k \alpha_j\varepsilon_{ijk}P_k .$$

The spin operators, in contrast, commute with **P**, but not necessarily with the various matrices. We work these out explicitly.

$$\left[S_i, \alpha_j\right] = \tfrac{1}{2}\hbar\left[\Sigma_i, \alpha_j\right] = \tfrac{1}{2}\hbar\left[\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}\begin{pmatrix}\sigma_j & 0\\ 0 & -\sigma_j\end{pmatrix} - \begin{pmatrix}\sigma_j & 0\\ 0 & -\sigma_j\end{pmatrix}\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}\right]$$

$$= \tfrac{1}{2}\hbar\begin{pmatrix}\left[\sigma_i, \sigma_j\right] & 0\\ 0 & -\left[\sigma_i, \sigma_j\right]\end{pmatrix} = i\hbar\sum_k \varepsilon_{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & -\sigma_k\end{pmatrix} = i\hbar\sum_k \varepsilon_{ijk}\alpha_k ,$$

$$\left[S_i, \beta\right] = \tfrac{1}{2}\hbar\left[\Sigma_i, \beta\right] = \tfrac{1}{2}\hbar\left[\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} - \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}\right] = 0.$$

We therefore have

$$\left[S_i, H\right] = \left[S_i, c\boldsymbol{\alpha}\cdot\mathbf{P} + mc^2\beta\right] = \sum_j \left[S_i, c\alpha_j P_j\right] = i\hbar c\sum_j\sum_k \varepsilon_{ijk}\alpha_k P_j ,$$

A more succinct version of these two formulas would be $\left[\mathbf{L}, H\right] = i\hbar c\boldsymbol{\alpha}\times\mathbf{P}$ and $\left[\mathbf{S}, H\right] = i\hbar c\mathbf{P}\times\boldsymbol{\alpha}$

c) [5] Show that $\left[\mathbf{J}, H\right] = 0$, where $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

We have

$$\left[J_i, H\right] = i\hbar c \sum_j \sum_k \alpha_j \varepsilon_{ijk} P_k + i\hbar c \sum_j \sum_k \varepsilon_{ijk} \alpha_k P_j = i\hbar c \sum_j \sum_k \varepsilon_{ijk} \left(\alpha_j P_k + \alpha_k P_j\right).$$

Exactly how you finish the reasoning depends on how you think about it. Perhaps the easiest way to see this is to interchange the indices *j* and *k* on the final term, which then yields

$$\left[J_i, H\right] = i\hbar c \sum_j \sum_k \left(\varepsilon_{ijk} + \varepsilon_{ikj}\right) \alpha_j P_k$$

Then the anti-symmetry of ε_{ijk} assures us that this vanishes, so $\left[\mathbf{J}, H\right] = 0$.