Physics 742 – Graduate Quantum Mechanics 2

Solutions to Chapter 17

1. [15] In class, we quantized the free electromagnetic field. In homework, you will quantize the free massive scalar field, with classical energy

$$E=\tfrac{1}{2}\int d^3\mathbf{r}\left\{\dot\phi^2(\mathbf{r},t)+c^2\left[\nabla\phi(\mathbf{r},t)\right]^2+\mu^2\phi^2(\mathbf{r},t)\right\}$$

This problem differs from the electromagnetic field in that: (i) there is no such thing as gauge choice; (ii) the field $\phi(\mathbf{r},t)$ is not a vector field; it doesn't have components, and (iii) there is a new term $\mu^2\phi^2$, unlike anything you've seen before.

a) [3] Write such a classical field $\phi(\mathbf{r},t)$ in terms of Fourier modes $\phi_\mathbf{k}(t)$. What is the relationship between $\phi_\mathbf{k}(t)$ and $\phi_{-\mathbf{k}}(t)$?

As for the electromagnetic field, we write $\phi(\mathbf{r},t)$ as

$$\phi(\mathbf{r},t)=\sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}}\phi_\mathbf{k}(t)$$

To make sure the field is real, we need $\sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}}\phi_\mathbf{k}(t)=\sum_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{r}}\phi_\mathbf{k}^*(t)$. If we let $\mathbf{k}\to-\mathbf{k}$ on the left side, we see that we need $\sum_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{r}}\phi_{-\mathbf{k}}(t)=\sum_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{r}}\phi_\mathbf{k}^*(t)$, which can be satisfied if

$$\phi_\mathbf{k}^*(t)=\phi_{-\mathbf{k}}(t).$$

b) [7] Substitute your expression for $\phi(\mathbf{r},t)$ into the expression for *E*. Work in finite volume *V* and do as many integrals and sums as possible.

$$\begin{aligned}
E&=\tfrac{1}{2}\sum_\mathbf{k}\sum_{\mathbf{k}'}\int d^3\mathbf{r}\left\{\dot\phi_\mathbf{k}(t)e^{i\mathbf{k}\cdot\mathbf{r}}\dot\phi_{\mathbf{k}'}(t)e^{i\mathbf{k}'\cdot\mathbf{r}}+c^2\nabla\left[\phi_\mathbf{k}(t)e^{i\mathbf{k}\cdot\mathbf{r}}\right]\cdot\nabla\left[\phi_{\mathbf{k}'}(t)e^{i\mathbf{k}'\cdot\mathbf{r}}\right]+\mu^2\phi_\mathbf{k}(t)e^{i\mathbf{k}\cdot\mathbf{r}}\phi_{\mathbf{k}'}(t)e^{i\mathbf{k}'\cdot\mathbf{r}}\right\}\\
&=\tfrac{1}{2}\sum_\mathbf{k}\sum_{\mathbf{k}'}\int e^{i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}}d^3\mathbf{r}\left[\dot\phi_\mathbf{k}(t)\dot\phi_{\mathbf{k}'}(t)+c^2i^2\mathbf{k}\cdot\mathbf{k}'\phi_\mathbf{k}(t)\phi_{\mathbf{k}'}(t)+\mu^2\phi_\mathbf{k}(t)\phi_{\mathbf{k}'}(t)\right]\\
&=\tfrac{1}{2}V\sum_\mathbf{k}\sum_{\mathbf{k}'}\delta_{\mathbf{k}+\mathbf{k}',0}\left[\dot\phi_\mathbf{k}(t)\dot\phi_{\mathbf{k}'}(t)+c^2i^2\mathbf{k}\cdot\mathbf{k}'\phi_\mathbf{k}(t)\phi_{\mathbf{k}'}(t)+\mu^2\phi_\mathbf{k}(t)\phi_{\mathbf{k}'}(t)\right]\\
&=\tfrac{1}{2}V\sum_\mathbf{k}\left[\dot\phi_\mathbf{k}(t)\dot\phi_{-\mathbf{k}}(t)+c^2\mathbf{k}^2\phi_\mathbf{k}(t)\phi_{-\mathbf{k}}(t)+\mu^2\phi_\mathbf{k}(t)\phi_{-\mathbf{k}}(t)\right]\\
&=\tfrac{1}{2}V\sum_\mathbf{k}\left[\dot\phi_\mathbf{k}(t)\dot\phi_\mathbf{k}^*(t)+\left(c^2\mathbf{k}^2+\mu^2\right)\phi_\mathbf{k}(t)\phi_\mathbf{k}^*(t)\right]=V\sum_{\mathbf{k}>0}\left[\dot\phi_\mathbf{k}(t)\dot\phi_\mathbf{k}^*(t)+\left(c^2\mathbf{k}^2+\mu^2\right)\phi_\mathbf{k}(t)\phi_\mathbf{k}^*(t)\right]
\end{aligned}$$

c) [3] Restrict the sum using *only* positive values of k. Argue that you now have a sum of classical complex harmonic oscillators. What is the formula for $\omega_{\mathbf{k}}$, the frequency for each of these oscillators?

I have already restricted the sum appropriately. If we define

$$\omega_{\mathbf{k}} = \sqrt{c^2\mathbf{k}^2 + \mu^2}$$

then this energy is the same as

$$E = V\sum_{\mathbf{k}>0}\left[\dot{\phi}_{\mathbf{k}}(t)\dot{\phi}^*_{\mathbf{k}}(t) + \omega^2_{\mathbf{k}}\phi_{\mathbf{k}}(t)\phi^*_{\mathbf{k}}(t)\right],$$

which is nothing more than the formula for a bunch of uncoupled harmonic oscillators, each having mass V and angular frequency $\omega_{\mathbf{k}}$. We are now ready to quantize the theory.

d) [2] Reinterpret *H* as a Hamiltonian, and quantize the resulting theory. Find an expression for the Hamiltonian in terms of creation and annihilation operators.

The Hamiltonian can now be written in a single step,

$$H = \sum_{\mathbf{k}>0}\hbar\omega_{\mathbf{k}}\left(a^\dagger_{+,\mathbf{k}}a_{+,\mathbf{k}} + a^\dagger_{-,\mathbf{k}}a_{-,\mathbf{k}} + 1\right)$$

This can be best rewritten by defining $a_{\mathbf{k}}$ for all **k**, both positive and negative, as for the EM field, and then H can be rewritten as

$$H = \sum_{\mathbf{k}}\hbar\omega_{\mathbf{k}}\left(a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \tfrac{1}{2}\right)$$

It is easy to see from this that the energy of the state with wave number **k** is just $\hbar\omega_{\mathbf{k}} = \sqrt{c^2\hbar^2\mathbf{k}^2 + \hbar^2\mu^2}$. Comparing this with the formula we would expect for a massive particle, $E_{\mathbf{k}} = \sqrt{c^2(\hbar\mathbf{k})^2 + m^2c^4}$ we see that this is the appropriate Hamiltonian for a free massive particle with mass $mc^2 = \hbar\mu$.

2. [20] How do we create the classical analog of a plane wave quantum mechanically? Naively, you simply use a large number of quanta.

a) [10] Suppose the EM field is in the quantum state $|n,\mathbf{q},\tau\rangle$, where *n* is large. Find the expectation value of the electric $\langle\mathbf{E}(\mathbf{r})\rangle$ and magnetic fields $\langle\mathbf{B}(\mathbf{r})\rangle$ for this quantum state. For definiteness, choose $\mathbf{q}=q\hat{\mathbf{z}}$ and $\boldsymbol{\varepsilon}_{\mathbf{q}\sigma}=\hat{\mathbf{x}}$

We start by writing

$$\langle n,\mathbf{q},\tau|\mathbf{E}(\mathbf{r})|n,\mathbf{q},\tau\rangle=\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}i\langle n,\mathbf{q},\tau|\left(e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}-e^{-i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}a^\dagger_{\mathbf{k}\sigma}\right)|n,\mathbf{q},\tau\rangle$$

Now, in the sum, some of the terms have annihilation operators. Since there is only one type of photon present, these will yield $a_{\mathbf{k}\sigma}|n,\mathbf{q},\tau\rangle=0$ unless $\mathbf{k}=\mathbf{q}$ and $\sigma=\tau$. Similarly, for the creation operators, we can let them act on the left to yield $\langle n,\mathbf{q},\tau|a^\dagger_{\mathbf{k}\sigma}=0$ for most of the terms. Therefore there is only one term in the sum, and we have

$$\langle\mathbf{E}(\mathbf{r})\rangle=\langle n,\mathbf{q},\tau|\mathbf{E}(\mathbf{r})|n,\mathbf{q},\tau\rangle=i\sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\langle n,\mathbf{q},\tau|\left(e^{i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}_{\mathbf{q}\tau}a_{\mathbf{q}\tau}-e^{-i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}a^\dagger_{\mathbf{q}\tau}\right)|n,\mathbf{q},\tau\rangle$$

$$=i\sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\left(e^{i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}_{\mathbf{q}\tau}\langle n,\mathbf{q},\tau|a_{\mathbf{q}\tau}|n,\mathbf{q},\tau\rangle-e^{-i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}\langle n,\mathbf{q},\tau|a^\dagger_{\mathbf{q}\tau}|n,\mathbf{q},\tau\rangle\right)$$

The first matrix element is $\langle n,\mathbf{q},\tau|a_{\mathbf{q}\tau}|n,\mathbf{q},\tau\rangle=\sqrt{n}\langle n,\mathbf{q},\tau|n-1,\mathbf{q},\tau\rangle=0$. The second is the Hermitian conjugate of this, and therefore also zero, so $\langle\mathbf{E}(\mathbf{r})\rangle=0$. Similarly,

$$\langle\mathbf{B}(\mathbf{r})\rangle=\langle n,\mathbf{q},\tau|\mathbf{B}(\mathbf{r})|n,\mathbf{q},\tau\rangle=\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}i\mathbf{k}\times\langle n,\mathbf{q},\tau|\left(e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}-e^{-i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}a^\dagger_{\mathbf{k}\sigma}\right)|n,\mathbf{q},\tau\rangle$$

$$=\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}i\mathbf{q}\times\left(e^{i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}_{\mathbf{q}\tau}\langle n,\mathbf{q},\tau|a_{\mathbf{q}\tau}|n,\mathbf{q},\tau\rangle-e^{-i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}\langle n,\mathbf{q},\tau|a^\dagger_{\mathbf{q}\tau}|n,\mathbf{q},\tau\rangle\right)$$

Obviously, this vanishes as well, so

$$\langle\mathbf{E}(\mathbf{r})\rangle=\langle\mathbf{B}(\mathbf{r})\rangle=0.$$

This bears little resemblance to our classical concept of an electromagnetic wave, since there is no average field.

b) [10] Now try a coherent state, given by $\left|\psi_c\right\rangle = e^{-|c|^2/2}\sum_{n=0}^{\infty}\left(c^n/\sqrt{n!}\right)\left|n,\mathbf{q},\tau\right\rangle$, where c is an arbitrary complex number. Once again, find the expectation value of the electric and magnetic field. Coherent states were described in section 5B.

Once again, only one term will enter the sum, because there is only one type of photon present (though now we aren't sure how many photons there are). We therefore find

$$\left\langle \mathbf{E}(\mathbf{r})\right\rangle = i\sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\left(e^{i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}_{\mathbf{q}\tau}\left\langle\psi_c\right|a_{\mathbf{q}\tau}\left|\psi_c\right\rangle - e^{-i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}\left\langle\psi_c\right|a^\dagger_{\mathbf{q}\tau}\left|\psi_c\right\rangle\right)$$

$$\left\langle \mathbf{B}(\mathbf{r})\right\rangle = \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_q}}i\mathbf{q}\times\left(e^{i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}_{\mathbf{q}\tau}\left\langle\psi_c\right|a_{\mathbf{q}\tau}\left|\psi_c\right\rangle - e^{-i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}\left\langle\psi_c\right|a^\dagger_{\mathbf{q}\tau}\left|\psi_c\right\rangle\right)$$

We now need to work out these two matrix elements (which are complex conjugates of each other). Following computations similar to those in section 5D, we see that

$$\begin{aligned} a_{\mathbf{q}\tau}\left|\psi_c\right\rangle &= e^{-|c|^2/2}\sum_{n=0}^{\infty}\frac{c^n}{\sqrt{n!}}a_{\mathbf{q}\tau}\left|n,\mathbf{q},\tau\right\rangle = e^{-|c|^2/2}\sum_{n=0}^{\infty}\frac{c^n}{\sqrt{n!}}\sqrt{n}\left|n-1,\mathbf{q},\tau\right\rangle \\ &= e^{-|c|^2/2}\sum_{n=1}^{\infty}\frac{c^n}{\sqrt{(n-1)!}}\left|n-1,\mathbf{q},\tau\right\rangle = e^{-|c|^2/2}\sum_{n=0}^{\infty}\frac{c^{n+1}}{\sqrt{n!}}\left|n,\mathbf{q}\tau\right\rangle = c\left|\psi_c\right\rangle . \end{aligned}$$

Assuming our state is properly normalized (which it is), we then find

$$\left\langle\psi_c\right|a_{\mathbf{q}\tau}\left|\psi_c\right\rangle = \left\langle\psi_c\right|c\left|\psi_c\right\rangle = c ,$$

and therefore

$$\left\langle\psi_c\right|a^\dagger_{\mathbf{q}\tau}\left|\psi_c\right\rangle = c^* .$$

We therefore have

$$\left\langle \mathbf{E}(\mathbf{r})\right\rangle = i\sqrt{\frac{\hbar\omega_q}{2\varepsilon_0 V}}\left(e^{i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}_{\mathbf{q}\tau}c - e^{-i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}c^*\right) = -\hat{\mathbf{x}}\sqrt{\frac{2\hbar cq}{\varepsilon_0 V}}\,\mathrm{Im}\left(ce^{iqz}\right),$$

$$\left\langle \mathbf{B}(\mathbf{r})\right\rangle = \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_q}}i\mathbf{q}\times\left(e^{i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}_{\mathbf{q}\tau}c - e^{-i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}c^*\right) = -\hat{\mathbf{y}}\sqrt{\frac{2\hbar q}{\varepsilon_0 Vc}}\,\mathrm{Im}\left(ce^{iqz}\right).$$

This closely resembles our concept of a plane wave, with perpendicular electric and magnetic fields oscillating like sines or cosines (depending on whether c is real or imaginary).