Solutions to Chapter 17

3. [15] Suppose we measure the instantaneous electric field using a probe of finite size, so that we actually measure $\mathbf{E}_f(\mathbf{r}) \equiv \int \mathbf{E}(\mathbf{r}+\mathbf{s}) f(\mathbf{s})\, d^3\mathbf{s}$, where $f(\mathbf{s}) = \pi^{-3/2} a^{-3} e^{-\mathbf{s}^2/a^2}$, where *a* is the characteristic size of the probe. For the vacuum state, find the expectation value of $\langle \mathbf{E}_f(\mathbf{r}) \rangle$ and $\langle \mathbf{E}_f^2(\mathbf{r}) \rangle$. You should take the infinite volume limit, and make sure your answer is independent of *V*.

I will follow the work of section 17G. We start by writing out $\mathbf{E}_f(\mathbf{r})$ in more detail:

$$\mathbf{E}_f(\mathbf{r}) = \int d^3\mathbf{s} f(\mathbf{s}) \mathbf{E}(\mathbf{r}+s) = \frac{1}{a^3 \pi^{\frac{3}{2}}} \int d^3\mathbf{s} e^{-\mathbf{s}^2/a^2} \sum_{\mathbf{k},\sigma} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} i \left(e^{i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}\cdot\mathbf{s}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} a_{\mathbf{k},\sigma} - e^{-i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}\cdot\mathbf{s}} \boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma} a^\dagger_{\mathbf{k},\sigma} \right).$$

We can do the **s** integrals in Cartesian coordinates with the help of (A.28) to yield

$$\int d^3\mathbf{s} e^{-\mathbf{s}^2/a^2} e^{i\mathbf{k}\cdot\mathbf{s}} = \left(a\sqrt{\pi}\right)^3 \exp\left[\tfrac{1}{4}(ik_x)^2 a^2 + \tfrac{1}{4}(ik_y)^2 a^2 + \tfrac{1}{4}(ik_y)^2 a^2\right] = a^3 \pi^{\frac{3}{2}} \exp\left(-\tfrac{1}{4}\mathbf{k}^2 a^2\right).$$

Hence we have

$$\mathbf{E}_f(\mathbf{r}) = \sum_{\mathbf{k},\sigma} e^{-\mathbf{k}^2 a^2/4} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} i \left(e^{i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} a_{\mathbf{k},\sigma} - e^{-i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma} a^\dagger_{\mathbf{k},\sigma} \right).$$

When this acts on the vacuum state, the annihilation part vanishes, so we have

$$\mathbf{E}_f(\mathbf{r})|0\rangle = -i \sum_{\mathbf{k},\sigma} e^{-\mathbf{k}^2 a^2/4} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} e^{-i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma} a^\dagger_{\mathbf{k},\sigma} |0\rangle = -i \sum_{\mathbf{k},\sigma} e^{-\mathbf{k}^2 a^2/4} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} e^{-i\mathbf{k}\cdot\mathbf{r}} \boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma} |1,\mathbf{k},\sigma\rangle.$$

It is then trivial that $\langle \mathbf{E}_f(\mathbf{r}) \rangle = 0$, but

$$\langle \mathbf{E}_f^2(\mathbf{r}) \rangle = \left| \mathbf{E}_f(\mathbf{r})|0\rangle \right|^2 = \frac{i(-i)\hbar}{2\varepsilon_0 V} \sum_{\mathbf{k}\sigma} \sum_{\mathbf{k}'\sigma'} e^{-\mathbf{k}^2 a^2/4} e^{-\mathbf{k}'^2 a^2/4} \sqrt{\omega_k \omega_{k'}} e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{k}'\cdot\mathbf{r}} \left(\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \cdot \boldsymbol{\varepsilon}^*_{\mathbf{k}'\sigma'} \right) \langle 1,\mathbf{k},\sigma | 1,\mathbf{k}',\sigma' \rangle$$

$$= \frac{\hbar}{2\varepsilon_0 V} \sum_{\mathbf{k}\sigma} \sum_{\mathbf{k}'\sigma'} e^{-\mathbf{k}^2 a^2/4} e^{-\mathbf{k}'^2 a^2/4} \sqrt{\omega_k \omega_{k'}} e^{i\mathbf{k}\cdot\mathbf{r} - i\mathbf{k}'\cdot\mathbf{r}} \left(\boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \cdot \boldsymbol{\varepsilon}^*_{\mathbf{k}'\sigma'} \right) \delta_{\mathbf{k},\mathbf{k}'} \delta_{\sigma,\sigma'} = \frac{\hbar}{2\varepsilon_0 V} \sum_{\mathbf{k}\sigma} e^{-\mathbf{k}^2 a^2/2} \omega_k.$$

Summing over polarizations just yields a factor of two, and replace $\omega_k = ck$. We now take the infinite volume limit, which turns the sum into an integral. We have

$$\langle \mathbf{E}_f^2(\mathbf{r}) \rangle = \frac{\hbar c}{\varepsilon_0} \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-\mathbf{k}^2 a^2/2} k = \frac{\hbar c}{\varepsilon_0} \frac{4\pi}{(2\pi)^3} \int_0^\infty k^2 dk\, e^{-k^2 a^2/2} k = \frac{\hbar c}{\pi^2 \varepsilon_0 a^4} \int_0^\infty \left(\tfrac{1}{2} a^2 k^2\right) e^{-\frac{1}{2}a^2 k^2} d\left(\tfrac{1}{2} a^2 k^2\right)$$

$$= \frac{\hbar c}{\pi^2 \varepsilon_0 a^4}.$$

As expected, the fluctuations get bigger the smaller the probe, and become infinite as $a \to 0$.