Solutions to Chapter 17

4. [25] Define, for the electric and magnetic field, the annihilation and creation parts as

$$\mathbf{E}_-(\mathbf{r}) = i\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}}a_{\mathbf{k}\sigma}, \qquad \mathbf{E}_+(\mathbf{r}) = -i\sum_{\mathbf{k}\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger_{\mathbf{k}\sigma},$$

$$\mathbf{B}_-(\mathbf{r}) = i\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}}a_{\mathbf{k}\sigma}, \quad \mathbf{B}_+(\mathbf{r}) = -i\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\mathbf{k}\times\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger_{\mathbf{k}\sigma}.$$

It should be obvious that $\mathbf{E}(\mathbf{r}) = \mathbf{E}_+(\mathbf{r}) + \mathbf{E}_-(\mathbf{r})$ and $\mathbf{B}(\mathbf{r}) = \mathbf{B}_+(\mathbf{r}) + \mathbf{B}_-(\mathbf{r})$.

(a) [9] Define the normal-ordered energy density as

$$\tilde{u}(\mathbf{r}) \equiv \tfrac{1}{2}\varepsilon_0\left\{\left[\mathbf{E}_+^2(\mathbf{r}) + 2\mathbf{E}_+(\mathbf{r})\cdot\mathbf{E}_-(\mathbf{r}) + \mathbf{E}_-^2(\mathbf{r})\right] + c^2\left[\mathbf{B}_+^2(\mathbf{r}) + 2\mathbf{B}_+(\mathbf{r})\cdot\mathbf{B}_-(\mathbf{r}) + \mathbf{B}_-^2(\mathbf{r})\right]\right\}$$

Prove that this normal-ordered energy density differs from the usual definition by a constant, *i.e.*, that the difference contains no operators (the constant will be infinite).

The definitions are given such that $\mathbf{E}_-(\mathbf{r})$ contains the "annihilation" portion of $\mathbf{E}$, and $\mathbf{E}_+(\mathbf{r})$ contains the "creation": portion, and similarly for the magnetic field. It is clear that $\mathbf{E}(\mathbf{r}) = \mathbf{E}_+(\mathbf{r}) + \mathbf{E}_-(\mathbf{r})$ and $\mathbf{B}(\mathbf{r}) = \mathbf{B}_+(\mathbf{r}) + \mathbf{B}_-(\mathbf{r})$. Therefore

$$u(\mathbf{r}) = \tfrac{1}{2}\varepsilon_0\left\{\left[\mathbf{E}_+(\mathbf{r}) + \mathbf{E}_-(\mathbf{r})\right]^2 + c^2\left[\mathbf{B}_+(\mathbf{r}) + \mathbf{B}_-(\mathbf{r})\right]^2\right\}$$

$$= \tfrac{1}{2}\varepsilon_0\left\{\begin{array}{l}\left[\mathbf{E}_+^2(\mathbf{r}) + \mathbf{E}_+(\mathbf{r})\cdot\mathbf{E}_-(\mathbf{r}) + \mathbf{E}_-(\mathbf{r})\cdot\mathbf{E}_+(\mathbf{r}) + \mathbf{E}_-^2(\mathbf{r})\right] \\ +c^2\left[\mathbf{B}_+^2(\mathbf{r}) + \mathbf{B}_+(\mathbf{r})\cdot\mathbf{B}_-(\mathbf{r}) + \mathbf{B}_-(\mathbf{r})\cdot\mathbf{B}_+(\mathbf{r}) + \mathbf{B}_-^2(\mathbf{r})\right]\end{array}\right\}$$

Subtracting, we see that

$$u(\mathbf{r}) - \tilde{u}(\mathbf{r}) = \tfrac{1}{2}\varepsilon_0\left[\mathbf{E}_-(\mathbf{r})\cdot\mathbf{E}_+(\mathbf{r}) - \mathbf{E}_+(\mathbf{r})\cdot\mathbf{E}_-(\mathbf{r}) + c^2\mathbf{B}_-(\mathbf{r})\cdot\mathbf{B}_+(\mathbf{r}) - c^2\mathbf{B}_+(\mathbf{r})\cdot\mathbf{B}_-(\mathbf{r})\right]$$

$$= \tfrac{1}{2}\varepsilon_0\frac{-i^2\hbar}{2\varepsilon_0 V}\sum_{\mathbf{k}\sigma}\sum_{\mathbf{k}'\sigma'}e^{i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}'\cdot\mathbf{r}}\left[\sqrt{\omega_k\omega_{k'}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\cdot\boldsymbol{\varepsilon}^*_{\mathbf{k}'\sigma'} + c^2\left(\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\right)\cdot\left(\mathbf{k}'\times\boldsymbol{\varepsilon}^*_{\mathbf{k}'\sigma'}\right)\big/\sqrt{\omega_k\omega_{k'}}\right]$$
$$\times\left(a_{\mathbf{k}\sigma}a^\dagger_{\mathbf{k}'\sigma'} - a^\dagger_{\mathbf{k}'\sigma'}a_{\mathbf{k}\sigma}\right)$$

$$= \frac{\hbar}{4V}\sum_{\mathbf{k}\sigma}\sum_{\mathbf{k}'\sigma'}e^{i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}'\cdot\mathbf{r}}\left[\sqrt{\omega_k\omega_{k'}}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\cdot\boldsymbol{\varepsilon}^*_{\mathbf{k}'\sigma'} + c^2\left(\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\right)\cdot\left(\mathbf{k}'\times\boldsymbol{\varepsilon}^*_{\mathbf{k}'\sigma'}\right)\big/\sqrt{\omega_k\omega_{k'}}\right]\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$$

$$= \frac{\hbar}{4V}\sum_{\mathbf{k}\sigma}\left[\omega_k\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\cdot\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma} + c^2\left(\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}\right)\cdot\left(\mathbf{k}\times\boldsymbol{\varepsilon}^*_{\mathbf{k}\sigma}\right)\big/\omega_k\right] = \frac{\hbar}{4V}\sum_{\mathbf{k}\sigma}\left(\omega_k + c^2k^2/\omega_k\right)$$

$$= \frac{\hbar c}{2V}\sum_{\mathbf{k}\sigma}k = \hbar c\int\frac{d^3\mathbf{k}}{(2\pi)^3} = \frac{\hbar c}{2\pi^2}\int_0^\infty k^3\,dk\,.$$

As advertised, the difference is not an operator and independent of $\mathbf{r}$, but it is infinite.

(b) [2] Prove that the expectation value of this operator for the vacuum is zero.

This can be done easily using reasoning. Every term in $\tilde{u}(\mathbf{r})$ has either an annihilation operator on the right or a creation operator on the left. Since $a_{\mathbf{k}\sigma}|0\rangle = 0 = \langle 0|a^\dagger_{\mathbf{k}\sigma}$, every term yields zero, so $\langle 0|\tilde{u}(\mathbf{r})|0\rangle = 0$.

(c) [10] Consider the quantum state $|\psi\rangle = \frac{\sqrt{8}}{3}|0\rangle + \frac{1}{3}|2,\mathbf{q},\tau\rangle$; *i.e.*, a quantum superposition of the vacuum and a two photon state with wave number q and polarization τ. To keep things simple, let the polarization $\boldsymbol{\varepsilon}_{\mathbf{q}\tau}$ be real. Work out the expectation value $\langle\tilde{u}(\mathbf{r})\rangle$ for this quantum state.

We start by writing

$$\langle\psi|\tilde{u}(\mathbf{r})|\psi\rangle = \tfrac{8}{9}\langle 0|\tilde{u}(\mathbf{r})|0\rangle + \tfrac{\sqrt{8}}{9}\langle 0|\tilde{u}(\mathbf{r})|2,\mathbf{q},\tau\rangle + \tfrac{\sqrt{8}}{9}\langle 2,\mathbf{q},\tau|\tilde{u}(\mathbf{r})|0\rangle + \tfrac{1}{9}\langle 2,\mathbf{q},\tau|\tilde{u}(\mathbf{r})|2,\mathbf{q},\tau\rangle$$

The first term vanishes, and the second and third terms are complex conjugates of each other. If you look, for example, at the second term, it is obvious that we must have two annihilation terms, so the only parts of $\tilde{u}(\mathbf{r})$ that contribute will be $\mathbf{E}_-^2(\mathbf{r})$ and $\mathbf{B}_-^2(\mathbf{r})$. Furthermore, in the resulting double sum, only those terms that annihilate the photon with momentum $\mathbf{q}$ and polarization τ will contribute. It follows that

$$\begin{aligned}
\langle 0|\tilde{u}(\mathbf{r})|2,\mathbf{q},\tau\rangle &= \tfrac{1}{2}\varepsilon_0\langle 0|\left[\mathbf{E}_-^2(\mathbf{r}) + c^2\mathbf{B}_-^2(\mathbf{r})\right]|2,\mathbf{q},\tau\rangle \\
&= \tfrac{1}{2}\varepsilon_0 i^2\left[\frac{\hbar\omega_q}{2\varepsilon_0 V}\boldsymbol{\varepsilon}_{\mathbf{q}\tau}^2 e^{2i\mathbf{q}\cdot\mathbf{r}} + c^2\frac{\hbar}{2\varepsilon_0 V\omega_q}\left(\mathbf{q}\times\boldsymbol{\varepsilon}_{\mathbf{q}\tau}\right)^2 e^{2i\mathbf{q}\cdot\mathbf{r}}\right]\langle 0|a_{\mathbf{q}\tau}^2|2,\mathbf{q},\tau\rangle \\
&= -\frac{\hbar}{4V}e^{2i\mathbf{q}\cdot\mathbf{r}}\left(\omega_q + \frac{c^2q^2}{\omega_q}\right)\sqrt{2} = -\frac{\hbar cq}{\sqrt{2}V}e^{2i\mathbf{q}\cdot\mathbf{r}}\ .
\end{aligned}$$

It follows that

$$\langle 2,\mathbf{q},\tau|\tilde{u}(\mathbf{r})|0\rangle = \langle 0|\tilde{u}(\mathbf{r})|2,\mathbf{q},\tau\rangle^* = -\frac{\hbar cq}{\sqrt{2}V}e^{-2i\mathbf{q}\cdot\mathbf{r}}$$

For the last term, it is clear we must look at the cross terms, $\mathbf{E}_+(\mathbf{r})\cdot\mathbf{E}_-(\mathbf{r})$ and $\mathbf{B}_+(\mathbf{r})\cdot\mathbf{B}_-(\mathbf{r})$, so that we have

$$\langle 2,\mathbf{q},\tau|\tilde{u}(\mathbf{r})|2,\mathbf{q},\tau\rangle = \varepsilon_0 \langle 2,\mathbf{q},\tau|\left[\mathbf{E}_+(\mathbf{r})\cdot\mathbf{E}_-(\mathbf{r}) + c^2\mathbf{B}_+(\mathbf{r})\cdot\mathbf{B}_-(\mathbf{r})\right]|2,\mathbf{q},\tau\rangle$$

$$= \varepsilon_0\left[\frac{\hbar\omega_q}{2\varepsilon_0 V}\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau}\cdot\boldsymbol{\varepsilon}_{\mathbf{q}\tau} + c^2\frac{\hbar(\mathbf{q}\times\boldsymbol{\varepsilon}^*_{\mathbf{q}\tau})\cdot(\mathbf{q}\times\boldsymbol{\varepsilon}_{\mathbf{q}\tau})}{2\varepsilon_0 V\omega_q}\right]\langle 2,\mathbf{q},\tau|a^\dagger_{\mathbf{q}\tau}a_{\mathbf{q}\tau}|2,\mathbf{q},\tau\rangle$$

$$= \left(\frac{\hbar\omega_q}{2V} + c^2\frac{\hbar q^2}{2V\omega_q}\right)2 = \frac{2\hbar cq}{V}.$$

Substituting it all in, we have

$$\langle\psi|\tilde{u}(\mathbf{r})|\psi\rangle = \frac{\sqrt{8}}{9}\left(-\frac{\hbar cq}{V\sqrt{2}}\right)\left(e^{2i\mathbf{q}\cdot\mathbf{r}} + e^{-2i\mathbf{q}\cdot\mathbf{r}}\right) + \frac{1}{9}\left(\frac{2\hbar cq}{V}\right) = \frac{\hbar cq}{9V}\left[2 - 4\cos(2\mathbf{q}\cdot\mathbf{r})\right]$$

(d) [4] Sketch $\langle\tilde{u}(\mathbf{r})\rangle$ as a function of $\mathbf{q}\cdot\mathbf{r}$. Note that it is sometimes negative (less energy than the vacuum!). Find its integral over space, and check that it does, however, have total energy positive.

Obviously, it goes negative at a variety of points, including the origin. However, if we integrate it over the volume of our universe, the cosine term goes away, and the integral yields a factor of V on the constant term, for a volume integral of $\frac{2}{9}\hbar cq$.