Physics 742 – Graduate Quantum Mechanics 2

Solutions to Chapter 18

1. [10] An electron is trapped in a 3D harmonic oscillator potential, $H = \mathbf{P}^2/2m + \frac{1}{2}m\omega_0^2\mathbf{R}^2$. It is in the quantum state $\left|n_x, n_y, n_z\right\rangle = \left|2,1,0\right\rangle$.

(a) [5] Calculate every non-vanishing matrix element of the form $\left\langle n_x', n_y', n_z'\right|\mathbf{R}\left|2,1,0\right\rangle$, where the final state is lower in energy than the initial state.

The three coordinate operators can be written in the form $R_i = \sqrt{\hbar/2m\omega_0}\left(a_i + a_i^\dagger\right)$, so

$$\begin{aligned}\mathbf{R}\left|210\right\rangle &= \sqrt{\frac{\hbar}{2m\omega_0}}\left[\hat{\mathbf{x}}\left(a_x + a_x^\dagger\right) + \hat{\mathbf{y}}\left(a_y + a_y^\dagger\right) + \hat{\mathbf{z}}\left(a_z + a_z^\dagger\right)\right]\left|210\right\rangle \\ &= \sqrt{\frac{\hbar}{2m\omega_0}}\left[\hat{\mathbf{x}}\sqrt{2}\left|110\right\rangle + \hat{\mathbf{x}}\sqrt{3}\left|310\right\rangle + \hat{\mathbf{y}}\left|200\right\rangle + \hat{\mathbf{y}}\sqrt{2}\left|220\right\rangle + \hat{\mathbf{z}}\left|211\right\rangle\right].\end{aligned}$$

We are interested in decay, which implies we want a lower energy state, so there are only two possible final states that work

$$\left\langle 110\right|\mathbf{R}\left|210\right\rangle = \hat{\mathbf{x}}\sqrt{\frac{\hbar}{m\omega_0}} \quad \text{and} \quad \left\langle 200\right|\mathbf{R}\left|210\right\rangle = \hat{\mathbf{y}}\sqrt{\frac{\hbar}{2m\omega_0}}.$$

(b) [5] Calculate the decay rate $\Gamma\left(210 \to n_x', n_y', n_z'\right)$ for this decay in the dipole approximation for every possible final state, and find the corresponding branching ratios.

The decay rate is given by $\Gamma\left(I \to F\right) = \frac{4}{3}\alpha\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2/c^2$. In each case, the frequency difference is simply ω_0, so we have

$$\Gamma\left(2,1,0 \to 1,1,0\right) = \frac{4\alpha\omega_0^3}{3c^2}\frac{\hbar}{m\omega_0} = \frac{4\alpha\hbar\omega_0^2}{3mc^2}, \quad \Gamma\left(2,1,0 \to 2,0,0\right) = \frac{4\alpha\omega_0^3}{3c^2}\frac{\hbar}{2m\omega_0} = \frac{2\alpha\hbar\omega_0^2}{3mc^2}.$$

The total decay rate is $\Gamma = \dfrac{2\alpha\hbar\omega_0^2}{mc^2}$. The branching ratios are

$$BR\left(2,1,0 \to 1,1,0\right) = \frac{4\alpha\hbar\omega_0^2}{3mc^2}\cdot\frac{mc^2}{2\alpha\hbar\omega_0^2} = \frac{2}{3}, \quad BR\left(2,1,0 \to 2,0,0\right) = \frac{2\alpha\hbar\omega_0^2}{3mc^2}\cdot\frac{mc^2}{2\alpha\hbar\omega_0^2} = \frac{1}{3}.$$

2. [15] A hydrogen atom is initially in a 3d state, specifically, $\left|n,l,m\right\rangle = \left|3,2,+2\right\rangle$.

(a) [9] Find all non-zero matrix elements of the form $\langle n',l',m'|\mathbf{R}|3,2,+2\rangle$, where $n'<n$. Which state(s) will it decay into?

The position operator is a rank one tensor, so by the Wigner-Eckart theorem, the final value of l' must be in the range $l'=l+1,l,l-1$, or $l'=3$, 2, or 1. But since $n'<3$, and $l'<n'$, the only possibility is $l'=1$, which in turn implies $n'=2$. Furthermore, if we look at the matrix elements of the various components of **R**, which in spherical tensor notation would be $R_q^{(1)}$, then $\langle 2,1,m'|R_q^{(1)}|3,2,+2\rangle$ is only non-vanishing for $m'+q=2$, which has the unique solution $m'=q=1$. So it *must* decay *only* to the state $|2,1,+1\rangle$.

We now need the non-zero matrix elements $\langle 2,1,1|\mathbf{R}|3,2,+2\rangle$, which we work out directly:

$$\langle 2,1,1|\mathbf{R}|3,2,+2\rangle=\int d^3\mathbf{r}R_{21}(r)Y_2^1(\theta,\phi)^*R_{32}(r)Y_2^2(\theta,\phi)r(\hat{\mathbf{x}}\sin\theta\cos\phi+\hat{\mathbf{y}}\sin\theta\sin\phi+\hat{\mathbf{z}}\cos\phi)$$

$$=-\frac{3\sqrt{5}}{16\pi}\int_0^\infty R_{21}(r)R_{32}(r)r^3dr\left[\int_0^{2\pi}(\hat{\mathbf{x}}\cos\phi+\hat{\mathbf{y}}\sin\phi)e^{i\phi}d\phi\int_0^\pi\sin^5\theta d\theta\right.$$
$$\left.+\hat{\mathbf{z}}\int_0^{2\pi}e^{i\phi}d\phi\int_0^\pi\sin^4\theta\cos\theta d\theta\right]$$

$$=-\frac{2^{11}3^4}{5^7}a_0(\hat{\mathbf{x}}+i\hat{\mathbf{y}}).$$

I used my hydrogen Maple routine to do the r and θ integral; the ϕ integrals are π, $i\pi$, and 0 respectively.

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> simplify(int(radial(3,2)*radial(2,1)*r^3,r=0..infinity)
  *int(sin(theta)^5,theta=0..Pi)*3*sqrt(5)/16);
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(b) [6] Calculate the decay rate in s^{-1}.

We start with our standard expression, namely

$$\Gamma(I\to F)=\frac{4\alpha\omega_{IF}^3|\mathbf{r}_{FI}|^2}{3c^2}=\frac{4\cdot2^{22}\cdot3^8\alpha\omega_{IF}^3a_0^2}{3\cdot5^{14}c^2}\left(|-1|^2+|-i|^2\right)=\frac{2^{25}\cdot3^7\alpha\omega_{IF}^3a_0^2}{5^{14}c^2}.$$

The frequency is given by the difference in energy between the two states, namely

$$\omega_{IF}=\frac{\varepsilon_I-\varepsilon_F}{\hbar}=\frac{1}{\hbar}\left(-\frac{mc^2\alpha^2}{2\cdot9}+\frac{mc^2\alpha^2}{2\cdot4}\right)=\frac{5mc^2\alpha^2}{2^3\cdot3^2\hbar}.$$

The Bohr radius is $a_0=\hbar/\alpha mc$. Substituting these expressions into the rate equation, we have

$$\Gamma(I\to F)=\frac{2^{25}\cdot3^7\alpha}{5^{14}c^2}\left(\frac{5mc^2\alpha^2}{2^3\cdot3^2\hbar}\right)^3\left(\frac{\hbar}{\alpha mc}\right)^2=\frac{3\cdot2^{16}\alpha^5mc^2}{5^{11}\hbar}=\frac{3\cdot2^{16}\cdot511{,}000\text{ eV}}{5^{11}(137.036)^5(6.582\times10^{-16}\text{ eV}\cdot\text{s})}$$
$$=6.469\times10^7\text{ s}^{-1}.$$