Physics 742 – Graduate Quantum Mechanics 2

Solutions to Chapter 18

3. [15] An electron is trapped in a 3D harmonic oscillator potential, $H = \mathbf{P}^2/2m + \frac{1}{2}m\omega_0^2\mathbf{R}^2$. It is in the quantum state $\left|n_x, n_y, n_z\right\rangle = \left|0,0,2\right\rangle$. It is going to decay *directly* into the ground state $\left|0,0,0\right\rangle$

(a) [1] Convince yourself that it cannot go there via the electric dipole transition. It can, however, go there via the electric quadrupole transition.

The position operators **R** only raise or lower one of the three eigenvalues, and hence $\left\langle 0,0,0\right|\mathbf{R}\left|0,0,2\right\rangle$ vanishes.

(b) [3] Calculate every non-vanishing matrix element of the form $\left\langle 0,0,0\right|R_iR_j\left|0,0,2\right\rangle$.

There is only one non-vanishing component, namely

$$\left\langle 0,0,0\right|ZZ\left|0,0,2\right\rangle = \frac{\hbar}{2m\omega_0}\left\langle 0,0,0\right|a_z^2\left|0,0,2\right\rangle = \frac{\hbar}{\sqrt{2}m\omega_0}$$

(c) [7] Calculate the polarized differential decay rate $d\Gamma_{\text{pol}}\left(002 \to 000\right)/d\Omega$ for this decay. This will require, among many other things, converting a sum to an integral in the infinite volume limit.

As found in the lecture notes, the decay rate for electric quadrupole is given by

$$\frac{d\Gamma}{d\Omega} = \frac{\alpha\omega_{IF}^3}{8\pi c^2}\left|\sum_j \left\langle\phi_F\right|\left(\mathbf{k}\cdot\mathbf{R}_j\right)\left(\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^*\cdot\mathbf{R}_j\right)\left|\phi_I\right\rangle\right|^2 = \frac{\alpha\omega_{IF}^3}{8\pi c^2}\left|k_z\varepsilon_z^*\left\langle 0,0,0\right|Z^2\left|0,0,2\right\rangle\right|^2 = \frac{\alpha\hbar^2\omega_{IF}^3k_z^2\left|\varepsilon_z\right|^2}{16\pi c^2m^2\omega_0^2}.$$

The energy difference between the states $\left|0,0,2\right\rangle$ and $\left|0,0,0\right\rangle$ is $2\hbar\omega_0$, and therefore $\omega_{IF} = 2\omega_0$, and $k_z = k\cos\theta = 2\omega_0\cos\theta/c$. Substituting this in, we have

$$\frac{d\Gamma}{d\Omega} = \frac{\alpha\hbar^2 8\omega_0^3}{16\pi c^2m^2\omega_0^2}\cdot\frac{4\omega_0^2\cos^2\theta}{c^2}\left|\varepsilon_z\right|^2 = \frac{2\alpha\hbar^2\omega_0^3}{\pi c^4m^2}\cos^2\theta\left|\varepsilon_z\right|^2.$$

(d) [4] Sum it over polarizations and integrate it over angles to determine the total decay rate $\Gamma\left(002 \to 000\right)$.

If we sum it over polarizations, the only polarization that will contribute is the one partially along the z-axis, for which we find $\boldsymbol{\varepsilon}_z = \sin\theta$, so we have

$$\frac{d\Gamma}{d\Omega} = \frac{2\alpha\hbar^2\omega_0^3}{\pi m^2 c^4}\cos^2\theta\sin^2\theta\,.$$

Integrating this over ϕ just gives a factor of 2π, and then the remaining integral yields

$$\Gamma = \frac{4\alpha\hbar^2\omega_0^3}{m^2 c^4}\int_0^\pi \cos^2\theta\sin^2\theta\sin\theta d\theta = \frac{16\alpha\hbar^2\omega_0^3}{15m^2 c^4}\,.$$