Physics 742 – Graduate Quantum Mechanics 2

Solutions to Chapter 18

5. [20] An electron is trapped in a 3D harmonic oscillator potential, $H = \mathbf{P}^2/2m + \frac{1}{2}m\omega_0^2\mathbf{R}^2$**. It is in the ground state** $|n_x, n_y, n_z\rangle = |0,0,0\rangle$**. Photons are fired at the electron with frequency** $\omega \ll \omega_0$**, going in the *z*-direction, and polarized in the *x*-direction,** $\boldsymbol{\varepsilon} = \hat{\mathbf{x}}$**. Calculate the differential cross section** $d\sigma/d\Omega$ **(summed over final polarizations, not integrated), and the total cross section** σ**.**

The amplitude for the scattering, according to the notes, Eq. (18.18) is

$$\mathcal{T}_{FI} = -\frac{e^2}{\varepsilon_0 V}\sum_m \frac{\omega\omega_{mI}}{\omega_{mI}^2 - \omega^2}\left(\boldsymbol{\varepsilon}_F \cdot \mathbf{r}_{mI}\right)^*\left(\boldsymbol{\varepsilon}_I \cdot \mathbf{r}_{mI}\right)$$

The initial photon is polarized in the *x*-direction, and therefore the relevant matrix element will be

$$\boldsymbol{\varepsilon}_I \cdot \mathbf{r}_{mI} = \langle m|X|000\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\langle m|\left(a_x + a_x^\dagger\right)|000\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\langle m|100\rangle$$

The only relevant intermediate state, then, is the $|100\rangle$ state. We therefore have

$$\mathcal{T}_{FI} = -\frac{e^2}{\varepsilon_0 V}\sum_m \frac{\omega\omega_{100,I}}{\omega_{100,I}^2 - \omega^2}\left(\boldsymbol{\varepsilon}_F \cdot \mathbf{r}_{100,I}\right)^*\left(\boldsymbol{\varepsilon}_I \cdot \mathbf{r}_{100,I}\right) = -\frac{e^2}{\varepsilon_0 V}\frac{\hbar}{2m\omega_0}\frac{\omega\omega_0}{\omega_0^2 - \omega^2}\varepsilon_{Fx}^* .$$

By Fermi's Golden rule, the rate for this transition is then

$$\Gamma = \frac{2\pi}{\hbar}\left|\mathcal{T}_{FI}\right|^2\delta\left(E_F - E_I\right) = \frac{2\pi}{\hbar}\left(\frac{\hbar e^2}{2\varepsilon_0 V m}\right)^2\frac{\omega^2}{\left(\omega_0^2 - \omega^2\right)^2}\left|\varepsilon_{Fx}\right|^2\delta\left(\hbar\omega_F - \hbar\omega\right).$$

We wish to sum this over polarizations and final wave numbers of the photon. The polarization vectors can be written as

$$\boldsymbol{\varepsilon}_1 = \left(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\right), \quad \boldsymbol{\varepsilon}_2 = \left(-\sin\phi, \cos\phi, 0\right),$$

so we then have

$$\begin{aligned}\Gamma &= \frac{2\pi e^4\omega^2}{4\varepsilon_0^2 V^2 m^2\left(\omega_0^2 - \omega^2\right)^2}\sum_{\mathbf{k}_F,\sigma}\left|\varepsilon_{Fx}\right|^2\delta\left(\omega_F - \omega\right) \\ &= \frac{\pi\left(4\pi\alpha c\hbar\right)^2\omega^2}{2Vm^2\left(\omega_0^2 - \omega^2\right)^2}\int\frac{d^3\mathbf{k}_F}{\left(2\pi\right)^3}\left(\cos^2\theta\cos^2\phi + \sin^2\phi\right)\delta\left(\omega_F - \omega\right),\end{aligned}$$

$$\Gamma = \frac{\alpha^2 \hbar^2 \omega^2}{Vm^2 c\left(\omega_0^2 - \omega^2\right)^2} \int \left(\cos^2\theta \cos^2\phi + \sin^2\phi\right) d\Omega_F \int_0^\infty \omega_F^2 d\omega_F \delta\left(\omega_F - \omega\right)$$

$$= \frac{\alpha^2 \hbar^2 \omega^4}{Vcm^2 \left(\omega_0^2 - \omega^2\right)^2} \int \left(\cos^2\theta \cos^2\phi + \sin^2\phi\right) d\Omega_F .$$

The rate is still inversely proportional to the volume. As usual, we convert this to a cross-section by dividing by the number density (which means multiplying by V) and dividing by the relative speed (c), so we have

$$\sigma = \frac{\alpha^2 \hbar^2 \omega^4}{c^2 m^2 \left(\omega_0^2 - \omega^2\right)^2} \int \left(\cos^2\theta \cos^2\phi + \sin^2\phi\right) d\Omega_F .$$

The unintegrated version of this is the differential cross-section, or we can go ahead and integrate it for the total cross-section.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \hbar^2 \omega^4}{c^2 m^2 \left(\omega_0^2 - \omega^2\right)^2} \left(\cos^2\theta \cos^2\phi + \sin^2\phi\right), \quad \text{and} \quad \sigma = \frac{8\pi \alpha^2 \hbar^2 \omega^4}{3c^2 m^2 \left(\omega_0^2 - \omega^2\right)^2} .$$

As expected, there is a huge enhancement near resonance, when $\omega \to \omega_0$.