

Physics 742 – Graduate Quantum Mechanics 2
Solutions to Chapter 18

5. [20] An electron is trapped in a 3D harmonic oscillator potential, $H = \mathbf{P}^2/2m + \frac{1}{2}m\omega_0^2\mathbf{R}^2$. It is in the ground state $|n_x, n_y, n_z\rangle = |0, 0, 0\rangle$. Photons are fired at the electron with frequency $\omega \ll \omega_0$, going in the z -direction, and polarized in the x -direction, $\boldsymbol{\varepsilon} = \hat{x}$. Calculate the differential cross section $d\sigma/d\Omega$ (summed over final polarizations, not integrated), and the total cross section σ .

The amplitude for the scattering, according to the notes, Eq. (18.18) is

$$\mathcal{T}_{FI} = -\frac{e^2}{\varepsilon_0 V} \sum_m \frac{\omega \omega_{ml}}{\omega_{ml}^2 - \omega^2} (\boldsymbol{\varepsilon}_F \cdot \mathbf{r}_{ml})^* (\boldsymbol{\varepsilon}_I \cdot \mathbf{r}_{ml})$$

The initial photon is polarized in the x -direction, and therefore the relevant matrix element will be

$$\boldsymbol{\varepsilon}_I \cdot \mathbf{r}_{ml} = \langle m | X | 000 \rangle = \sqrt{\frac{\hbar}{2m\omega_0}} \langle m | (a_x + a_x^\dagger) | 000 \rangle = \sqrt{\frac{\hbar}{2m\omega_0}} \langle m | 100 \rangle$$

The only relevant intermediate state, then, is the $|100\rangle$ state. We therefore have

$$\mathcal{T}_{FI} = -\frac{e^2}{\varepsilon_0 V} \sum_m \frac{\omega \omega_{100,I}}{\omega_{100,I}^2 - \omega^2} (\boldsymbol{\varepsilon}_F \cdot \mathbf{r}_{100,I})^* (\boldsymbol{\varepsilon}_I \cdot \mathbf{r}_{100,I}) = -\frac{e^2}{\varepsilon_0 V} \frac{\hbar}{2m\omega_0} \frac{\omega \omega_0}{\omega_0^2 - \omega^2} \boldsymbol{\varepsilon}_{Fx}^*.$$

By Fermi's Golden rule, the rate for this transition is then

$$\Gamma = \frac{2\pi}{\hbar} |\mathcal{T}_{FI}|^2 \delta(E_F - E_I) = \frac{2\pi}{\hbar} \left(\frac{\hbar e^2}{2\varepsilon_0 V m} \right)^2 \frac{\omega^2}{(\omega_0^2 - \omega^2)^2} |\boldsymbol{\varepsilon}_{Fx}|^2 \delta(\hbar\omega_F - \hbar\omega).$$

We wish to sum this over polarizations and final wave numbers of the photon. The polarization vectors can be written as

$$\boldsymbol{\varepsilon}_1 = (\cos\theta \cos\phi, \cos\theta \sin\phi, -\sin\theta), \quad \boldsymbol{\varepsilon}_2 = (-\sin\phi, \cos\phi, 0),$$

so we then have

$$\begin{aligned} \Gamma &= \frac{2\pi e^4 \omega^2}{4\varepsilon_0^2 V^2 m^2 (\omega_0^2 - \omega^2)^2} \sum_{\mathbf{k}_F, \sigma} |\boldsymbol{\varepsilon}_{Fx}|^2 \delta(\omega_F - \omega) \\ &= \frac{\pi (4\pi\alpha c\hbar)^2 \omega^2}{2Vm^2 (\omega_0^2 - \omega^2)^2} \int \frac{d^3\mathbf{k}_F}{(2\pi)^3} (\cos^2\theta \cos^2\phi + \sin^2\phi) \delta(\omega_F - \omega), \end{aligned}$$

$$\begin{aligned}\Gamma &= \frac{\alpha^2 \hbar^2 \omega^2}{Vm^2 c (\omega_0^2 - \omega^2)^2} \int (\cos^2 \theta \cos^2 \phi + \sin^2 \phi) d\Omega_F \int_0^\infty \omega_F^2 d\omega_F \delta(\omega_F - \omega) \\ &= \frac{\alpha^2 \hbar^2 \omega^4}{Vcm^2 (\omega_0^2 - \omega^2)^2} \int (\cos^2 \theta \cos^2 \phi + \sin^2 \phi) d\Omega_F.\end{aligned}$$

The rate is still inversely proportional to the volume. As usual, we convert this to a cross-section by dividing by the number density (which means multiplying by V) and dividing by the relative speed (c), so we have

$$\sigma = \frac{\alpha^2 \hbar^2 \omega^4}{c^2 m^2 (\omega_0^2 - \omega^2)^2} \int (\cos^2 \theta \cos^2 \phi + \sin^2 \phi) d\Omega_F.$$

The unintegrated version of this is the differential cross-section, or we can go ahead and integrate it for the total cross-section.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \hbar^2 \omega^4}{c^2 m^2 (\omega_0^2 - \omega^2)^2} (\cos^2 \theta \cos^2 \phi + \sin^2 \phi), \quad \text{and} \quad \sigma = \frac{8\pi \alpha^2 \hbar^2 \omega^4}{3c^2 m^2 (\omega_0^2 - \omega^2)^2}.$$

As expected, there is a huge enhancement near resonance, when $\omega \rightarrow \omega_0$.