Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 1

3. [5] An electron in the ground state of hydrogen has, in spherical coordinates, the wave function $\psi(r,\theta,\phi) = Ne^{-r/a}$ where *N* and *a* are positive constants. Determine the normalization constant *N* and the probability that a measurement of the position will yield *r* > *a*. Don't forget you are working in three dimensions!

In three dimensions, working in spherical coordinates, the normalization condition is

$$1 = \iiint |\psi(\mathbf{r})|^2 d^3\mathbf{r} = \int_0^\infty r^2 dr \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi |\psi(\mathbf{r})|^2$$

In this case, the wave function depends only on *r*, so the inner two integrals are trivial.

$$1 = 4\pi N^2 \int_0^\infty r^2 e^{-2r/a} dr = 4\pi N^2 \left(-\tfrac{1}{2}r^2 a - \tfrac{1}{2}ra^2 - \tfrac{1}{4}a^3\right) e^{-2r/a}\Big|_0^\infty = \pi a^3 N^2 ,$$

$$N = 1\Big/\sqrt{\pi a^3}$$

$$P(r > a) = 4\pi N^2 \int_a^\infty r^2 e^{-2r/a} dr = \frac{4}{a^3}\left(-\tfrac{1}{2}r^2 a - \tfrac{1}{2}ra^2 - \tfrac{1}{4}a^3\right) e^{-2r/a}\Big|_a^\infty = 5e^{-2} = 67.67\%$$

4. [10] For each of the normalized wave functions given below, find the Fourier transform $\tilde{\psi}(k)$, and check that it satisfies the normalization condition $\int_{-\infty}^{\infty}\left|\tilde{\psi}(k)\right|^2 dk = 1$.

(a) [5] $\psi(x) = (A/\pi)^{1/4} \exp\left(iKx - \tfrac{1}{2}Ax^2\right)$

This turns into a Gaussian of the type you can find in Appendix A:

$$\tilde{\psi}(k) = \left(\frac{A}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp\left(iKx - \tfrac{1}{2}Ax^2\right)\exp(-ikx) = \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \exp\left[i(K-k)x - \tfrac{1}{2}Ax^2\right]$$

$$= \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2\pi}{A}} \exp\left\{\frac{\left[i(K-k)\right]^2}{4(A/2)}\right\} = (\pi A)^{-1/4} \exp\left[-(K-k)^2/2A\right].$$

We check this by simply seeing if the normalization works out:

$$\int_{-\infty}^{\infty}\left|\tilde{\psi}(k)\right|^2 dk = \frac{1}{\sqrt{\pi A}} \int_{-\infty}^{\infty} \exp\left[-(K-k)^2/A\right] dk = \frac{1}{\sqrt{\pi A}} \int_{-\infty}^{\infty} e^{-k^2/A} dk = \frac{1}{\sqrt{\pi A}} \sqrt{\frac{\pi}{1/A}} = 1.$$

(b) [5] $\psi(x) = \sqrt{\alpha}\exp(-\alpha|x|)$

We divide the integral into two pieces, then let $x \to -x$ on half of it. It is helpful to know $\int_0^{\infty} e^{-ax} dx = 1/a$ if a has a real positive part.

$$\tilde{\psi}(k) = \sqrt{\alpha} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\alpha|x|} e^{-ikx} = \sqrt{\frac{\alpha}{2\pi}} \left(\int_0^{\infty} e^{-\alpha x - ikx} dx + \int_{-\infty}^{0} e^{\alpha x - ikx} dx\right)$$

$$= \sqrt{\frac{\alpha}{2\pi}} \left(\int_0^{\infty} e^{-\alpha x - ikx} dx + \int_0^{\infty} e^{-\alpha x + ikx} dx\right) = \sqrt{\frac{\alpha}{2\pi}} \left(\frac{1}{\alpha + ik} + \frac{1}{\alpha - ik}\right) = \sqrt{\frac{\alpha}{2\pi}} \frac{2\alpha}{\alpha^2 + k^2}.$$

Once again, we check it, using the trig substitution $k = \alpha\tan\theta$ to complete the integral.

$$\int_{-\infty}^{\infty}\left|\tilde{\psi}(k)\right|^2 dk = \frac{4\alpha^3}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{\left(k^2 + \alpha^2\right)^2} = \frac{2\alpha^3}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\alpha \sec^2\theta d\theta}{\left(\alpha^2\tan^2\theta + \alpha^2\right)^2} = \frac{2}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^2\theta d\theta = 1.$$